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NON-DIFFUSIVE $N + 2$ DEGREE PETROV–GALERKIN METHODS FOR TWO-DIMENSIONAL TRANSIENT TRANSPORT COMPUTATIONS

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SUMMARY

A new non-diffusive Petrov–Galerkin type finite element method which uses test functions two polynomial degrees higher than the trial functions is developed for the transient convection dominated transport equation in two dimensions. The scheme uses bilinear quadrilateral finite elements for the spatial discretization and Crank–Nicolson finite differencing for the time integration. The standard product extension of very successful one-dimensional $N + 2$ degree upwinding functions to two dimensions is ineffective for general 2-D flow problems, especially at higher Courant numbers where cross-derivative truncation terms become important. Therefore effective $N + 2$ degree test functions are developed through an analysis by which the truncation error terms in the discrete nodal equation are eliminated up to fifth order. The new scheme is very effective for general 2-D flows over a wide Courant number range and eliminates the troublesome cross-derivative truncation terms. The scheme is simple and robust in that the upwinding coefficients are readily defined and only dependent on Courant number. Numerical examples illustrate the excellent behaviour of the new scheme.

INTRODUCTION

In a recent paper, Westerink and Shea[1] studied a new Petrov–Galerkin type finite element method for the solution of convection dominated transient transport problems in one dimension. This class of methods, originally proposed by Dick,[2] uses Lagrangian polynomials as basis functions and modified weighting functions which are two polynomial degrees higher than the basis functions. This new Petrov–Galerkin method has been designated as the $N + 2$ degree Petrov–Galerkin method, N being the degree of the polynomial used as the basis function. The work by Westerink and Shea[1] showed that $N + 2$ degree upwinded weighting functions used in conjunction with linear and quadratic trial functions dramatically improve the solutions for linear problems by effectively eliminating the space and especially time truncation errors. Comparisons with the results of standard Galerkin and traditional $N + 1$ degree Petrov–Galerkin methods indicate the superiority of $N + 2$ Petrov–Galerkin solutions which achieve perfect amplitude and substantially enhanced phase behaviour over a wide Courant number range. The purpose of this study is to extend the $N + 2$ degree Petrov–Galerkin method to two-dimensional transient convection dominated problems.

It is well known that the extension of successful one-dimensional solution techniques to two-dimensional finite elements poses considerable difficulties owing to the wide variety of additional cross terms which are present in a general two-dimensional problem. Classical $N + 1$ degree upwinding methods have been extended to two dimensions by simply taking the product of the one-dimensional upwind biased weighting functions.[3,4] It has been shown that such

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straightforward extensions generally lead to artificial crosswind diffusion, where excessive numerical diffusion is unnecessarily introduced in the direction perpendicular to the flow. The streamline upwind approach of Hughes and Brooks[5] and of Kelly *et al.*[6] which applies an optimal amount of artificial diffusion only in the direction of the flow, and the streamline upwinded Petrov–Galerkin scheme of Hughes and Brooks[7] and Brooks and Hughes,[8] which modify the test functions by a perturbation dependent on the velocity field and the derivative of the basis functions, successfully eliminate this problem. Another remedy against crosswind diffusion is the implementation of $N+1$ degree upwinding by weighting the upwinding coefficients on a given element side in proportion to the average direction cosines of the velocity vectors for that side. In this way, the artificial diffusion matrix becomes invariant with respect to the rotation of the co-ordinate system and hence any artificial crosswind diffusion is eliminated.

For time dependent problems, solely limiting the added artificial diffusion to the flow direction and improving the spatial discretization properties is found to be ineffective at improving the phase lag and numerical dispersion which stem from the additional complexities introduced by difficult time discretizations. Many recent works on the transient transport equation have therefore focused on improving the time discretization characteristics in conjunction with the use of effective spatial discretization methods. A number of successful schemes have emerged which competently handle general two-dimensional transient convection dominated problems. The Taylor–Galerkin method of Donea *et al.*[9] improves the time integration through Taylor series expanding the time derivative term such that second and third time derivatives are included. A recent technique developed by Yu and Heinrich[10,11] utilizes space–time finite elements geared towards improving the time integration. Tezduyar and Hughes[12] and Tezduyar and Ganjoo[13] have modified the perturbation terms for the streamline upwind/Petrov–Galerkin method so that they enhance the temporal discretization characteristics. Carey and Jiang[14] recently introduced a very promising least squares finite element scheme which has some similarities to both the Taylor–Galerkin method and streamline upwinding.

The straightforward extension of $N+2$ degree upwinding to two dimensions by taking the product of the one-dimensional $N+2$ degree biased weighting functions has not been successful for general two-dimensional flows.[2] It appears that significant cross-derivative truncation terms are not effectively eliminated. In particular, when the temporal discretization is difficult, this straightforward product extension of $N+2$ degree upwinding does not eliminate the large oscillations which follow directly behind a distribution transversely crossing a grid. Therefore, an $N+2$ degree upwinding scheme must be developed in two dimensions so that the cross-derivative truncation terms as well as uni-directional truncation terms can be effectively eliminated. This is done in the following sections by developing discretized equations and the truncation error for the most general complete third degree two-dimensional biased weighting function for bilinear quadrilateral basis functions and examining how the various two-dimensional upwinding terms can be used to effectively eliminate both uni-directional and cross type truncation errors. A detailed truncation analysis for the transient pure convection equation is presented in the Appendix.

GOVERNING EQUATIONS AND DISCRETIZATION DEVELOPMENT

We consider the time dependent linear two-dimensional convection–diffusion equation over a simply connected open region Ω with boundary Γ:

$$\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial x}\left(D_{xx}\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_{yy}\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial x}\left(D_{xy}\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial y}\left(D_{yx}\frac{\partial\phi}{\partial x}\right) \qquad (1)$$

where ϕ is the transported quantity, u and v are the velocities in x and y directions and $D_{i,j}$'s are the physical diffusion coefficients in the respective directions. The spatial discretization of equation (1) is accomplished through the development of a standard weak weighted residual formulation. Weighted integration of equation (1) with the flux boundary error term and application of Green's theorem yields

$$\iint_{\Omega}\left\{\left(\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}\right)W+\left(D_{xx}\frac{\partial\phi}{\partial x}\frac{\partial W}{\partial x}+D_{yy}\frac{\partial\phi}{\partial y}\frac{\partial W}{\partial y}\right.\right.$$
$$\left.\left.+D_{xy}\frac{\partial\phi}{\partial y}\frac{\partial W}{\partial x}+D_{yx}\frac{\partial\phi}{\partial x}\frac{\partial W}{\partial y}\right)\right\}\mathrm{d}\Omega-\int_{\Gamma_N}\bar{q}_n W\,\mathrm{d}\Gamma_N=0 \quad (2)$$

where $\bar{q}_n$ is the prescribed boundary flux and W is the weighting function which will be thoroughly discussed in the next section.

The solution over the region Ω is approximated as

$$\phi=\sum_{j=1}^{n}\phi_j(t)\psi_j(x,y) \quad (3)$$

where $\phi_j(t)$ are undetermined, time dependent coefficients, and the $\psi_j(x, y)$ are the basis or shape functions associated with the nodes.

Substituting the approximating series (3) into the weak weighted residual formulation, (2), and some algebraic manipulation leads to the following system of spatially discretized ordinary differential equations:

$$\mathbf{M}\frac{\partial\boldsymbol{\phi}}{\partial t}+\mathbf{K}\boldsymbol{\phi}=\mathbf{P} \quad (4)$$

where $\mathbf{M}$, the mass matrix, is given as

$$\mathbf{M}=\sum_{\text{elem}}\iint_{\Omega^e}\psi_i W_j\,\mathrm{d}\Omega^e \quad (5)$$

and $\mathbf{K}$ is the stiffness matrix, which can be expressed as

$$\mathbf{K}=\sum_{\text{elem}}\iint_{\Omega^e}\left\{W_i\left(u\frac{\partial\psi_j}{\partial x}+v\frac{\partial\psi_j}{\partial y}\right)\right.$$
$$\left.+D_{xx}\frac{\partial W_i}{\partial x}\frac{\partial\psi_j}{\partial x}+D_{yy}\frac{\partial W_i}{\partial y}\frac{\partial\psi_j}{\partial y}+D_{xy}\frac{\partial W_i}{\partial x}\frac{\partial\psi_j}{\partial y}+D_{yx}\frac{\partial W_i}{\partial y}\frac{\partial\psi_j}{\partial x}\right\}\mathrm{d}\Omega^e \quad (6)$$

and $\mathbf{P}$ is the boundary flux vector which equals

$$\mathbf{P}=\int_{\Gamma_N}W_i\bar{q}_n\,\mathrm{d}\Gamma_N \quad (7)$$

The time discretization of equation (4) can be implemented through the use of a Crank–Nicolson finite difference scheme to obtain

$$\left[\mathbf{M}+\frac{\Delta}{2}\mathbf{K}^{n+1}\right]\boldsymbol{\phi}^{n+1}=\left[\mathbf{M}-\frac{\Delta}{2}\mathbf{K}^{n}\right]\boldsymbol{\phi}^{n}+\frac{1}{2}\mathbf{P}^{n+1}+\frac{1}{2}\mathbf{P}^{n} \quad (8)$$

where $n+1$ and n represent the future and current time levels, and Δ represents the time step.

GENERAL WEIGHTING FUNCTION DEVELOPMENT

We now consider (1) for the case with no diffusion. We define standard bilinear Lagrange polynomials as the trial functions:

$$\begin{aligned}\psi_1(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1-\eta)\\ \psi_2(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1-\eta)\\ \psi_3(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1+\eta)\\ \psi_4(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1+\eta)\end{aligned} \tag{9}$$

We bias these trial functions such that the weighting functions are defined as

$$W_i(\xi,\eta) = \psi_i(\xi,\eta) - F_i(\xi,\eta) \tag{10}$$

where $F_i(\xi,\eta)$ is the most general modifying function which is a complete cubic polynomial in two dimensions;

$$\begin{aligned}F_i(\xi,\eta) = {} & a_{i,1} + a_{i,2}\xi + a_{i,3}\eta + a_{i,4}\xi^2 + a_{i,5}\xi\eta + a_{i,6}\eta^2 + a_{i,7}\xi^3 + a_{i,8}\xi^2\eta\\ & + a_{i,9}\xi\eta^2 + a_{i,10}\eta^3 + a_{i,11}\xi^3\eta + a_{i,12}\xi^2\eta^2 + a_{i,13}\xi\eta^3\\ & + a_{i,14}\xi^3\eta^2 + a_{i,15}\xi^2\eta^3 + a_{i,16}\xi^3\eta^3\end{aligned} \tag{11}$$

where $a_{i,m}$ ($i = 1, \ldots, 4$ and $m = 1, \ldots, 16$) are unknown coefficients which must be determined.

Noting that $F_i(\xi,\eta)$ must be equal to zero at the element nodes, we eliminate four unknowns and after some algebraic manipulation we get

$$\begin{aligned}F_i(\xi,\eta) = {} & -a_{i,4}(1-\xi^2) - a_{i,6}(1-\eta^2) - a_{i,7}\xi(1-\xi^2) - a_{i,8}\eta(1-\xi^2)\\ & - a_{i,9}\xi(1-\eta^2) - a_{i,10}\eta(1-\eta^2) - a_{i,11}\xi\eta(1-\xi^2)\\ & - a_{i,12}(1-\xi^2\eta^2) - a_{i,13}\xi\eta(1-\eta^2) - a_{i,14}\xi(1-\xi^2\eta^2)\\ & - a_{i,15}\eta(1-\xi^2\eta^2) - a_{i,16}\xi\eta(1-\xi^2\eta^2)\end{aligned} \tag{12}$$

In equation (12), terms $a_{i,4}$ and $a_{i,6}$ are the one-dimensional $N+1$ degree terms. Terms with coefficients $a_{i,8}$, $a_{i,9}$ and $a_{i,12}$ simply correspond to cross product $N+1$ degree terms. For the standard product extension of one-dimensional $N+1$ degree upwinding functions to two dimensions, the weighting functions for bilinear elements equal

$$\begin{aligned}W_1(\xi,\eta) = {} & \psi_1(\xi,\eta) - (\tfrac{3}{8}\alpha_x - \tfrac{9}{16}\alpha_x\alpha_y)(1-\xi^2) - (\tfrac{3}{8}\alpha_y - \tfrac{9}{16}\alpha_x\alpha_y)(1-\eta^2)\\ & + \tfrac{3}{8}\alpha_x\eta(1-\xi^2) + \tfrac{3}{8}\alpha_y\xi(1-\eta^2) - \tfrac{9}{16}\alpha_x\alpha_y(1-\xi^2\eta^2)\end{aligned} \tag{13a}$$

$$\begin{aligned}W_2(\xi,\eta) = {} & \psi_2(\xi,\eta) + (\tfrac{3}{8}\alpha_x - \tfrac{9}{16}\alpha_x\alpha_y)(1-\xi^2) - (\tfrac{3}{8}\alpha_y + \tfrac{9}{16}\alpha_x\alpha_y)(1-\eta^2)\\ & - \tfrac{3}{8}\alpha_x\eta(1-\xi^2) - \tfrac{3}{8}\alpha_y\xi(1-\eta^2) + \tfrac{9}{16}\alpha_x\alpha_y(1-\xi^2\eta^2)\end{aligned} \tag{13b}$$

$$\begin{aligned}W_3(\xi,\eta) = {} & \psi_3(\xi,\eta) + (\tfrac{3}{8}\alpha_x + \tfrac{9}{16}\alpha_x\alpha_y)(1-\xi^2) + (\tfrac{3}{8}\alpha_y + \tfrac{9}{16}\alpha_x\alpha_y)(1-\eta^2)\\ & + \tfrac{3}{8}\alpha_x\eta(1-\xi^2) + \tfrac{3}{8}\alpha_y\xi(1-\eta^2) - \tfrac{9}{16}\alpha_x\alpha_y(1-\xi^2\eta^2)\end{aligned} \tag{13c}$$

$$\begin{aligned}W_4(\xi,\eta) = {} & \psi_4(\xi,\eta) - (\tfrac{3}{8}\alpha_x + \tfrac{9}{16}\alpha_x\alpha_y)(1-\xi^2) + (\tfrac{3}{8}\alpha_y - \tfrac{9}{16}\alpha_x\alpha_y)(1-\eta^2)\\ & - \tfrac{3}{8}\alpha_x\eta(1-\xi^2) - \tfrac{3}{8}\alpha_y\xi(1-\eta^2) + \tfrac{9}{16}\alpha_x\alpha_y(1-\xi^2\eta^2)\end{aligned} \tag{13d}$$

where α_x and α_y respectively represent the $N+1$ degree upwinding coefficients for the horizontal and vertical element sides. When these coefficients are set equal to the product of the standard uni-directional $N+1$ degree upwinding coefficient and the average direction cosines of the velocity vector for that side, it can be shown that the artificial diffusion matrix is invariant with respect to the rotation of the co-ordinate system and hence no artificial crosswind diffusion is introduced. We designate this scheme as $N+1$/SPE (Standard Product Extension) upwinding.

We also note that the terms in equation (12) with coefficients $a_{i,7}$ and $a_{i,10}$ are the $N+2$ degree biasing terms which have been shown to improve the solution drastically for one-dimensional cases. Terms with coefficients $a_{i,11}$, $a_{i,13}$ and $a_{i,16}$ correspond to cross product $N+2$ degree terms. The standard product extension of one-dimensional $N+2$ degree upwinding functions to two dimensions leads to the following set of weighting functions:

$$\begin{aligned} W_1(\xi,\eta) = \psi_1(\xi,\eta) &+ \tfrac{5}{16}\beta_x\xi(1-\xi^2) + \tfrac{5}{16}\beta_y\eta(1-\eta^2) - (\tfrac{5}{16}\beta_x - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\xi^2) \\ &- (\tfrac{5}{16}\beta_y - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\eta^2) - \tfrac{25}{64}\beta_x\beta_y\xi\eta(1-\xi^2\eta^2) \end{aligned} \tag{14a}$$

$$\begin{aligned} W_2(\xi,\eta) = \psi_2(\xi,\eta) &+ \tfrac{5}{16}\beta_x\xi(1-\xi^2) - \tfrac{5}{16}\beta_y\eta(1-\eta^2) + (\tfrac{5}{16}\beta_x - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\xi^2) \\ &+ (\tfrac{5}{16}\beta_y - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\eta^2) + \tfrac{25}{64}\beta_x\beta_y\xi\eta(1-\xi^2\eta^2) \end{aligned} \tag{14b}$$

$$\begin{aligned} W_3(\xi,\eta) = \psi_3(\xi,\eta) &- \tfrac{5}{16}\beta_x\xi(1-\xi^2) - \tfrac{5}{16}\beta_y\eta(1-\eta^2) - (\tfrac{5}{16}\beta_x - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\xi^2) \\ &- (\tfrac{5}{16}\beta_y - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\eta^2) - \tfrac{25}{64}\beta_x\beta_y\xi\eta(1-\xi^2\eta^2) \end{aligned} \tag{14c}$$

$$\begin{aligned} W_4(\xi,\eta) = \psi_4(\xi,\eta) &- \tfrac{5}{16}\beta_x\xi(1-\xi^2) + \tfrac{5}{16}\beta_y\eta(1-\eta^2) + (\tfrac{5}{16}\beta_x - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\xi^2) \\ &+ (\tfrac{5}{16}\beta_y - \tfrac{25}{64}\beta_x\beta_y)\xi\eta(1-\eta^2) + \tfrac{25}{64}\beta_x\beta_y\xi\eta(1-\xi^2\eta^2) \end{aligned} \tag{14d}$$

where β_x and β_y respectively represent the $N+2$ degree upwinding coefficients of the horizontal and vertical element sides. We shall designate this as $N+2$/SPE upwinding. As was indicated earlier, $N+2$/SPE upwinding is entirely unsuccessful in treating general two-dimensional problems with difficult temporal discretizations. This can readily be demonstrated using truncation error analysis.

We now proceed by examining how $N+2$ degree upwinding can be effectively implemented in two dimensions such that it successfully eliminates the troublesome cross-derivative truncation terms which appear at large Courant numbers. We start by discarding all the one-dimensional and cross product terms in equation (12) associated with $N+1$ degree upwinding. It can be shown by truncation error analysis that, as in the one-dimensional case, $N+1$ degree upwinding terms cannot solely eliminate the truncation error beyond third order unless all individual sets of $N+1$ degree terms are set to zero. With this consideration, our general $N+2$ upwinded weighting function reduces to

$$\begin{aligned} W_i(\xi,\eta) = \psi_i(\xi,\eta) &+ a_{i,7}\xi(1-\xi^2) + a_{i,10}\eta(1-\eta^2) + a_{i,11}\xi\eta(1-\xi^2) + a_{i,13}\xi\eta(1-\eta^2) \\ &+ a_{i,14}\xi(1-\xi^2\eta^2) + a_{i,15}\eta(1-\xi^2\eta^2) + a_{i,16}\xi\eta(1-\xi^2\eta^2) \end{aligned} \tag{15}$$

Using the shape functions given in equation (9) and the weighting function given in equation (15), the element matrices can easily be formed in terms of unknown coefficients $a_{i,m}$. By assembling these element matrices and substituting into equation (8), global equations for any non-boundary node (i,j) are obtained. Since these equations in this very general form are relatively long they will not be reproduced here. A truncation error analysis for the nodal point difference equation can be performed by Taylor series expanding the equation about node (i,j). We then consider the original form of our governing equation (1) and perform sequential substitutions and/or take spatial derivatives such that we express all time derivative terms with corresponding spatial derivative terms. Substituting these expressions into the Taylor series expanded form of the nodal difference equation and grouping of terms with equal order spatial derivative of ϕ yields the general form of truncation error for the $N+2$ upwinded finite element scheme with bilinear shape functions. The development of the new $N+2$ upwinded weighting function is then accomplished by eliminating the truncation errors associated with each term in the Taylor expansion up to fifth order derivatives by assigning the appropriate values to the variable coefficients in equation (15). The details of this analysis for the pure convection equation are given in the Appendix. After defining the new $N+2$ degree upwinding parameters such that

$$\hat{\beta}_x = 2C_x^2 \tag{16a}$$

$$\hat{\beta}_y = 2C_y^2 \tag{16b}$$

$$\hat{\beta}_{xy} = C_x C_y \tag{16c}$$

where $C_x = u\Delta/h$ and $C_y = v\Delta/k$, in which Δ is the time step and h and k are spatial discretization sizes, the final form of the weighting functions can be reproduced as

$$W_1(\xi,\eta) = \psi_1(\xi,\eta) + \tfrac{5}{16}\hat{\beta}_x\xi(1-\xi^2) + \tfrac{5}{16}\hat{\beta}_y\eta(1-\eta^2) + \tfrac{225}{64}\hat{\beta}_{xy}\xi\eta(1-\xi^2\eta^2) \tag{17a}$$

$$W_2(\xi,\eta) = \psi_2(\xi,\eta) - \tfrac{5}{16}\hat{\beta}_x\xi(1-\xi^2) + \tfrac{5}{16}\hat{\beta}_y\eta(1-\eta^2) + \tfrac{225}{64}\hat{\beta}_{xy}\xi\eta(1-\xi^2\eta^2) \tag{17b}$$

$$W_3(\xi,\eta) = \psi_3(\xi,\eta) - \tfrac{5}{16}\hat{\beta}_x\xi(1-\xi^2) - \tfrac{5}{16}\hat{\beta}_y\eta(1-\eta^2) + \tfrac{225}{64}\hat{\beta}_{xy}\xi\eta(1-\xi^2\eta^2) \tag{17c}$$

$$W_4(\xi,\eta) = \psi_4(\xi,\eta) + \tfrac{5}{16}\hat{\beta}_x\xi(1-\xi^2) - \tfrac{5}{16}\hat{\beta}_y\eta(1-\eta^2) + \tfrac{225}{64}\hat{\beta}_{xy}\xi\eta(1-\xi^2\eta^2) \tag{17d}$$

The coefficients as defined in (16) lead to a scheme which is formally fourth order accurate for all Courant numbers. It is noted that, for the case of uni-directional flow, this set of weighting functions reduces to its one-dimensional counterpart discussed by Westerink and Shea.[1] However, the addition of the sixth degree polynomial with coefficient $\hat{\beta}_{xy}$ will now allow for the very effective elimination of the troublesome cross-derivative truncation terms. We will refer to our newly developed scheme as $N+2$/TUSC (third order uni-directional and sixth order cross term polynomial weighting) upwinding.

As was the case for the one-dimensional scheme, the x and y uni-directional $N+2$/TUSC upwinding coefficients, $\hat{\beta}_x$ and $\hat{\beta}_y$, must be increased somewhat over the truncation error analysis values given by equations (16a) and (16b). Based on one-dimensional Fourier analysis, it can be shown that perfect phase propagation characteristics result for a given wavelength, λ, when the upwinding coefficient is selected by computing the smallest positive root of

$$\beta(\lambda/h, C) = \frac{1}{2\left[\cos\left(\frac{2\pi}{\lambda/h}\right) - 1\right]}\left\{ -8\cos\left(\frac{2\pi}{\lambda/h}\right) - 16 + \left[\cos\left(\frac{2\pi C}{\lambda/h}\right) \pm 1\right]\left[\frac{12C\sin\left(\frac{2\pi}{\lambda/h}\right)}{\sin\left(\frac{2\pi C}{\lambda/h}\right)}\right]\right\} \tag{18}$$

where $C = u\Delta/h$. For the longest wavelengths, $\lambda/h = \infty$, equation (18) reduces to $\beta = 2C^2$, which corresponds to the truncation error analysis result. For the shortest resolvable wavelengths, $\lambda/h = 2$, equation (18) reduces to $\beta = 2$. Based on our examination of phase portraits, β should be computed using λ/h values somewhere between 3·5 and 6·0. This β range significantly improves the phase curve for short and intermediate wavelengths, while not deteriorating the phase properties of somewhat longer wavelengths. For two-dimensional schemes, $\hat{\beta}_x$ and $\hat{\beta}_y$ should be computed using equation (18), substituting for C by C_x and C_y respectively and keeping the λ/h range the same as for the one-dimensional scheme. The cross upwinding coefficient does not require any adjustments from the truncation analysis result and $\hat{\beta}_{xy}$ values given by equation (16c) lead to optimal performance characteristics. We note that, for general two-dimensional flows, the $N+2$/TUSC upwinding scheme exhibits perfect amplitude behaviour and a remarkably improved phase portrait compared to the standard Bubnov–Galerkin solution. The extent of the phase improvement increases as Courant number increases up to $C = 1$.

For two-dimensional flows, unlike one-dimensional flows, the $N+2$/TUSC Petrov–Galerkin scheme as given in equation (17) modifies the elemental mass, convection and diffusion matrices. The change in the elemental mass matrix is due to both the uni-directional and cross upwinding terms and results in more weighting at the nodes in the direction of the flow. In the standard Bubnov–Galerkin solution however, the mass accumulation appears on the node associated with the elemental equation. The modification in the convection matrix is due to the uni-directional $N+2$/TUSC upwinding terms and occurs at the nodes in the direction of the flow only. The cross $\hat{\beta}_{xy}$ term has no contribution to the convection matrix. For the diffusion matrix, the unidirectional $N+2$/TUSC terms do not make any contribution whereas the cross $\hat{\beta}_{xy}$ term does. Numerical experimentation with diffusive problems suggests that the cross term contribution to the diffusion matrix may have an adverse effect on the solution, especially at high Courant numbers. Therefore, we recommend setting the value of cross upwinding coefficients to zero when differentiation of the upwind weighting functions is implemented (i.e. not allowing any cross term upwinding contribution to affect the diffusion matrix).

NUMERICAL EXAMPLES

The performance of our newly developed $N+2$/TUSC scheme is tested using a transversely translating concentration hill problem (e.g. Donea *et al.*,[9] Baker and Soliman[15]) and a rotating concentration hill problem (e.g. Baptista *et al.*[16]). For the first problem, results are compared for the standard Bubnov–Galerkin, traditional $N+1$/SPE and $N+2$/SPE degree Petrov–Galerkin schemes as given by equations (13) and (14) and our newly developed $N+2$/TUSC Petrov–Galerkin method. For the latter problem, comparisons are shown only between the standard Bubnov–Galerkin and $N+2$/TUSC Petrov–Galerkin schemes. The optimal values for the coefficients used in the $N+1$/SPE schemes are taken from the ranges specified by Westerink and Shea[1] and extended to two dimensions by multiplying them with the average direction cosines of velocity on the various element sides, as explained earlier. Numerical experimentation was used to determine the optimal $N+2$/SPE coefficient values. The uni-directional $N+2$/TUSC coefficients, $\hat{\beta}_x$ and $\hat{\beta}_y$, were computed using the extended two-dimensional version of equation (18) using various λ/h values which bracket the optimal solution. The cross $N+2$/TUSC upwinding coefficient, $\hat{\beta}_{xy}$, was always computed using equation (16c). The error criteria used to assess performance are listed in Table I. Error criteria include overall integral error (E1), relative peak damping (E2), the relative size of the maximum spurious oscillation (E3) and the phase shift of the distribution peak (E4).

The first test problem consists of a square domain with 35×35 nodes with a node to node distance equal to 200. A Gaussian type concentration distribution with standard deviation $\sigma = 264$ located in the left rear corner is given as the initial condition. Zero upstream boundary conditions are applied in both directions. The ambient flow field is defined through uniform

Table I. Error criteria for example problems[16]

Error E1: Integral measure of the overall error

$$E1 = \frac{1}{m(t)}\left[\int_0^L [\phi^{num}(x,y,t) - \phi^{ex}(x,y,t)]^2 \,dx\,dy\right]^{1/2}$$

Error E2: Point measure of the artificial damping of the numerical solution

$$E2 = \left|\frac{\phi^{ex}_{max}(t) - \phi^{num}_{max}(t)}{\phi^{ex}_{max}(t)}\right|$$

Error E3: Point measure of the maximum spurious oscillation in the numerical solution

$$E3 = \left|\frac{\phi^{num}_{max,neg}(t)}{\phi^{ex}_{max}(t)}\right|$$

Error E4: Point measure of the phase shift introduced in the numerical solution

$$E4 = \frac{r^{ex}_{max}(t) - r^{num}_{max}}{r^{ex}_{max}(t)}$$

where r_{max} is the distance travelled

Note: For the exact solutions all errors equal zero

Table II. Summary table for example problems and corresponding errors

Case-run	Fig	Weighting function	Weighting factor	E1	E2	E3	E4
1–1	2	Standard	—	0·000290	0·221455	0·117088	0·041667
1–2	3	$N+1$/SPE	$\alpha_x = \alpha_y = 0{\cdot}350$	0·000257	0·329017	0·040485	0·000000
1–3	4	$N+2$/SPE	$\beta_x = \beta_y = 0{\cdot}500$	0·000141	0·062912	0·038754	0·000000
1–4	—	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 0{\cdot}500$ ($\lambda/h = 3{\cdot}75$)	0·000149	0·059427	0·026930	0·000000
1–5	5	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 0{\cdot}450$ ($\lambda/h = 4{\cdot}00$)	0·000122	0·059371	0·019103	0·000000
1–6	—	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 0{\cdot}375$ ($\lambda/h = 4{\cdot}50$)	0·000100	0·068968	0·028173	0·000000
2–1	6	Standard	—	0·000760	0·354743	0·324367	0·041667
2–2	7	$N+1$/SPE	$\alpha_x = \alpha_y = 1{\cdot}414$	0·000599	0·526294	0·129309	0·041667
2–3	8	$N+2$/SPE	$\beta_x = \beta_y = 1{\cdot}300$	0·000626	0·442237	0·237918	0·000000
2–4	—	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 1{\cdot}411$ ($\lambda/h = 3{\cdot}75$)	0·000056	0·037373	0·012011	0·000000
2–5	9	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 1{\cdot}393$ ($\lambda/h = 4{\cdot}00$)	0·000055	0·041468	0·012339	0·000000
2–6	—	$N+2$/TUSC	$\hat\beta_x = \hat\beta_y = 1{\cdot}367$ ($\lambda/h = 4{\cdot}50$)	0·000060	0·048679	0·012375	0·000000
3–1	11	Standard	—	0·000710	0·330723	0·267363	0·014469
3–2	—	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 3{\cdot}75$)	0·000201	0·068292	0·040297	0·000000
3–3	12	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 4{\cdot}00$)	0·000181	0·072286	0·054349	0·000000
3–4	—	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 4{\cdot}50$)	0·000172	0·088640	0·079011	0·000000
4–1	14	Standard	—	0·000255	0·130486	0·202846	0·014469
4–2	—	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 3{\cdot}75$)	0·000062	0·007898	0·004751	0·000000
4–3	15	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 4{\cdot}00$)	0·000054	0·004681	0·003093	0·000000
4–4	—	$N+2$/TUSC	$\hat\beta_x \neq \hat\beta_y$ = varies ($\lambda/h = 4{\cdot}50$)	0·000045	0·003362	0·001941	0·000000

steady velocities $u = v = 0{\cdot}5$ in the x and y directions and zero physical diffusion is specified such that the concentration hill moves diagonally across the domain in pure convection. The exact solution at $t = 9600$ is shown in Figure 1.

Case 1 solves the defined transverse flow problem using relatively low Courant numbers such that $C_x = 0{\cdot}24$ and $C_y = 0{\cdot}24$. Error criteria values for this case are summarized in Table II. The standard Bubnov–Galerkin solution, Case 1–1 shown in Figure 2, exhibits a considerable drop in peak amplitude and trailing oscillations which propagate in two trains, each perpendicular to the grid lines. In addition there is a slight phase lag of the plume itself. These errors are entirely caused by numerical dispersion. Application of $N + 1$/SPE upwinding (with $\alpha_x = \alpha_y = 0{\cdot}35$, which corresponds to $\alpha = 0{\cdot}5$), Case 1–2 shown in Figure 3, eliminates the phase lag and trailing perpendicular oscillations which arise mainly from the spatial discretization, but also causes the peak to be further depressed due to numerical damping. Traditional application of $N + 2$/SPE upwinding, Case 1–3 shown in Figure 4, shows a much improved solution as compared to the standard Bubnov–Galerkin solution. Here the optimal upwinding coefficients were found to be $\beta_x = \beta_y = 0{\cdot}50$. Peak and overall accuracy have substantially improved, although reduced trailing oscillations caused by the spatial discretization still remain. Case 1–5 shown in Figure 5 indicates that further improvements in the quality of the solution are achieved when the $N + 2$/TUSC scheme specified with equation (17) is applied. However, $N + 2$/TUSC is only slightly superior to $N + 2$/SPE for this low Courant number case, as is indicated by the error criteria in Table II. This is due to the fact that cross-derivative type truncation terms are not very important at low Courant numbers. The $N + 2$/TUSC uni-directional upwinding parameters used for Case 1–5 ($\hat{\beta}_x = \hat{\beta}_y = 0{\cdot}450$) were computed using equation (18) with $C_x = C_y = 0{\cdot}24$ and $\lambda/h = 4{\cdot}0$, which gives the best overall solution when considering all the error criteria. However, sensitivity to the $\hat{\beta}_x$ and $\hat{\beta}_y$ values within the range $\lambda/h = 3{\cdot}5 \rightarrow 6$ is not dramatic. We note that the $\hat{\beta}_x$ and $\hat{\beta}_y$ values within this overall λ/h range are about 2 to 4 times larger than values given by the truncation error analysis prediction, equations (16a) and (16b). Numerical experimentation indicates that equation (16c) for the coefficient $\hat{\beta}_{xy}$ of the sixth degree cross term, determined from truncation error analysis, does give an optimum value. Finally, we note that the optimal uni-directional coefficients for both the $N + 2$/SPE and $N + 2$/TUSC schemes are nearly the same.

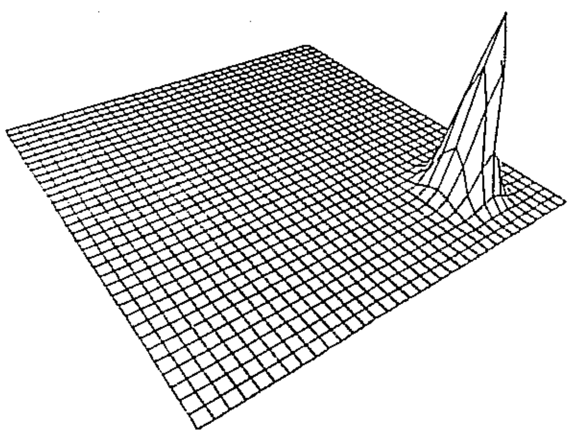

Figure 1. Analytical solution of a 2-D Gaussian plume travelling diagonally across the grid in pure convection at $t = 9600$ ($\sigma = 264$, $u = v = 0{\cdot}5$)

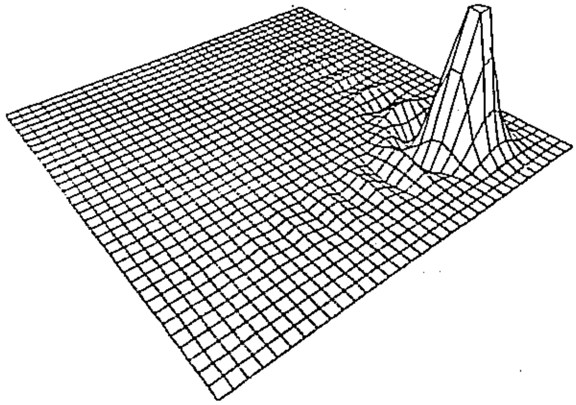

Figure 2. Case 1–1: Standard Bubnov–Galerkin solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the low Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}24$)

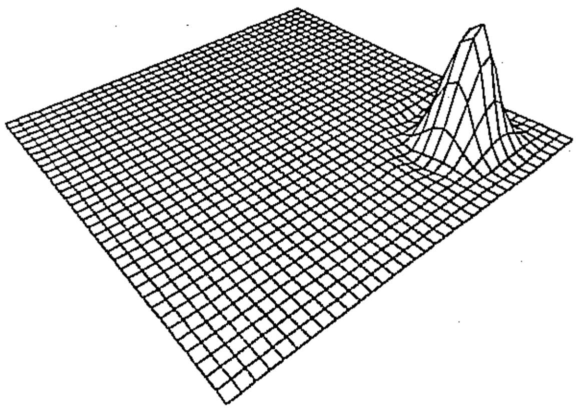

Figure 3. Case 1–2: $N + 1$/SPE Petrov–Galerkin ($\alpha_x = \alpha_y = 0{\cdot}35$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the low Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}24$)

Case 2 is identical to the problem defined for Case 1 with Courant number increased to $C_x = C_y = 0{\cdot}8$. Again error criteria values are listed in Table II. Case 2–1 in Figure 6 shows the standard Bubnov–Galerkin solution at this higher Courant number. The oscillations now trail the plume in a direction parallel to the flow. Furthermore, the plume is severely depressed and exhibits a pronounced phase lag, again due to numerical dispersion. These problems are primarily temporal difficulties and the associated cross-derivative type truncation terms are now important. As in the one-dimensional case, $N + 1$/SPE upwinding with a value of $\alpha_x = \alpha_y = 1{\cdot}41$ (which corresponds to $\alpha = 2{\cdot}0$) reduces the wiggles at the expense of severe over-damping of the solution, as is shown for Case 2–2 in Figure 7. Similarly, $N + 2$/SPE upwinding, using upwinding coefficients $\beta_x = \beta_y = 1{\cdot}30$, is not capable of eliminating the cross-truncation error terms which

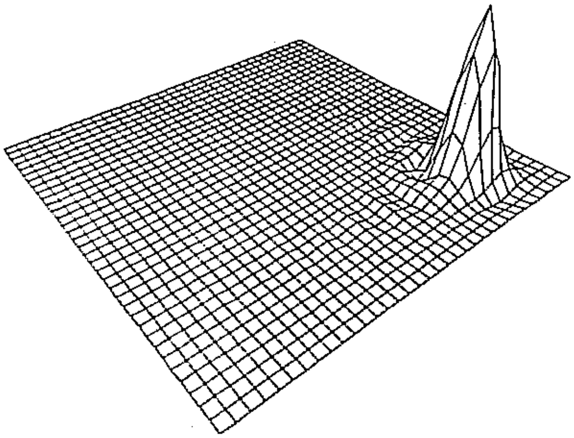

Figure 4. Case 1–3: $N + 2$/SPE Petrov–Galerkin ($\beta_x = \beta_y = 0{\cdot}50$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the low Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}24$)

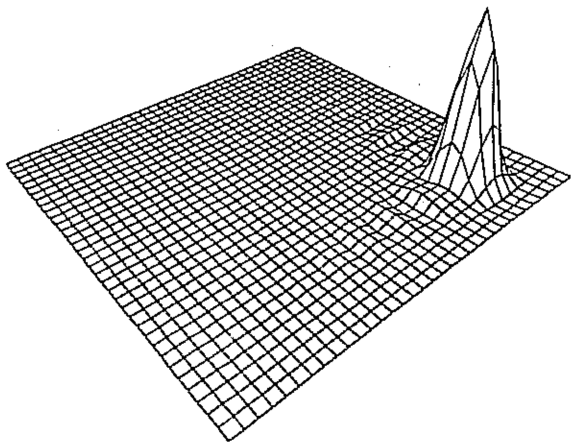

Figure 5. Case 1–5: $N + 2$/TUSC Petrov–Galerkin ($\bar{\beta}_x = \bar{\beta}_y = 0{\cdot}45$, $\bar{\beta}_{xy} = 0{\cdot}0576$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the low Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}24$)

introduce excessive phase errors into the solution, as is manifested by the trailing wiggles and the considerable drop in the peak for Case 2–3 shown in Figure 8. When our $N + 2$/TUSC upwinding scheme is used, the solution improves remarkably, as is shown for Case 2–5 in Figure 9. The peak amplitude shows only 4 per cent error and only very small trailing wiggles remain in the solution (1 per cent). The values for the uni-directional upwinding coefficients were computed using $\lambda/h = 4{\cdot}0$ and are equal to $\bar{\beta}_x = \bar{\beta}_y = 1{\cdot}39$. These optimal coefficient values are now only slightly higher than those predicted by truncation error analysis. The optimal value for the cross term coefficient, $\bar{\beta}_{xy} = 0{\cdot}8 \times 0{\cdot}8$, again remains the same as that determined by the truncation error analysis.

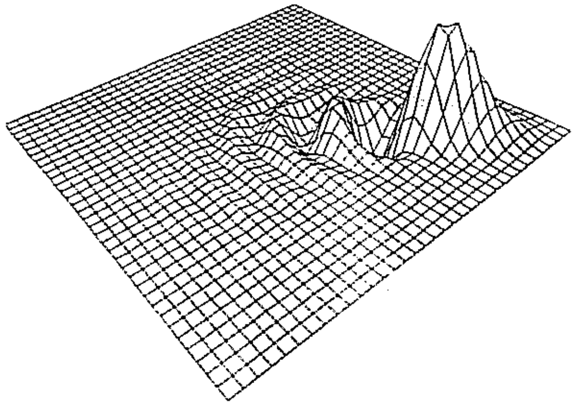

Figure 6. Case 2–1: Standard Bubnov–Galerkin solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the high Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}80$)

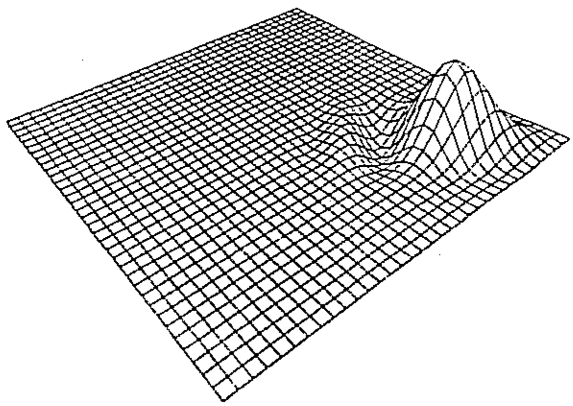

Figure 7. Case 2–2: $N + 1$/SPE Petrov–Galerkin ($\alpha_x = \alpha_y = 1{\cdot}41$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the high Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}80$)

The second test problem considers a Gaussian distribution transported in pure convection in a two-dimensional rotational flow field. The domain is identical to the square domain used for the transversely translating cone problem but a homogeneous essential boundary condition is now imposed everywhere on Γ.[16] The velocity field is given by $u = -\omega(y - 3400)$ and $v = \omega(x - 3400)$, where the origin of the co-ordinate system is located at the rear left corner of the domain. The constant angular velocity ω is taken as $2\pi/6000$ and the initial plume is located at $x_0 = 1200$ and $y_0 = 3400$. The time step is set to $\Delta = 60$, and thus the Courant number ranges between 0·0 and 1·07 over the domain and equals 0·70 at the distribution peak. Numerical solutions after one complete rotation are presented.

Case 3 solves the defined rotating flow problem for a narrow Gaussian distribution with $\sigma = 264$. The initial condition and final solution are shown in Figure 10. Error criteria values are

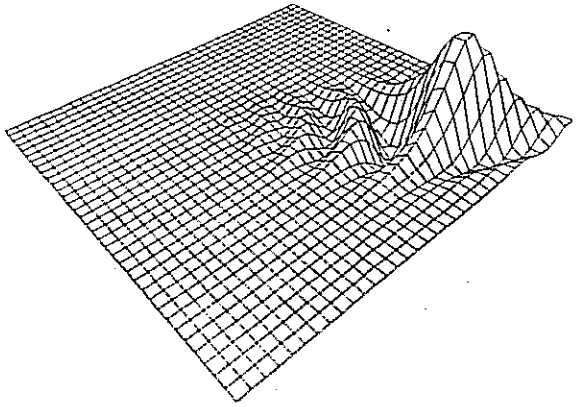

Figure 8. Case 2–3: $N + 2$/SPE Petrov–Galerkin ($\beta_x = \beta_y = 1{\cdot}30$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the high Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}80$)

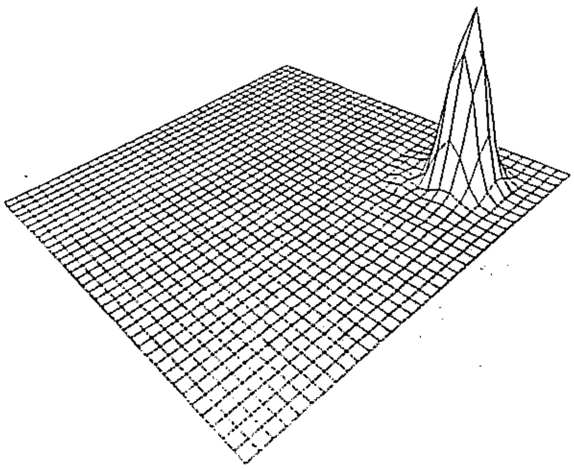

Figure 9. Case 2–5: $N + 2$/TUSC Petrov–Galerkin ($\hat{\beta}_x = \hat{\beta}_y = 1{\cdot}39$, $\hat{\beta}_{xy} = 0{\cdot}64$) solution of a 2-D Gaussian plume in pure convection at $t = 9600$ for the high Courant number flow ($\sigma = 264$, $C_x = C_y = 0{\cdot}80$)

listed in Table II. As is shown for Case 3–1 in Figure 11, the standard Bubnov–Galerkin scheme exhibits severe peak deterioration, very large wiggles trailing the distribution and a phase lag for the distribution itself. The use of the $N + 2$/TUSC method, Case 3–3 shown in Figure 12, dramatically improves the quality of the solution. The peak error has been reduced to 7 per cent while the largest dip has been reduced to 5 per cent. The uni-directional upwinding coefficients $\hat{\beta}_x$ and $\hat{\beta}_y$ were computed using $\lambda/h = 4{\cdot}0$. As is noted from Table II, a slightly improved solution, at least in terms of peak and dip errors, is obtained by using $\hat{\beta}_x$ and $\hat{\beta}_y$ values which correspond to a lower λ/h value.

Case 4 solves the same rotating flow problem for a wider Gaussian distribution with $\sigma = 400$. The initial condition and final solution after one complete rotation are shown in Figure 13. Error

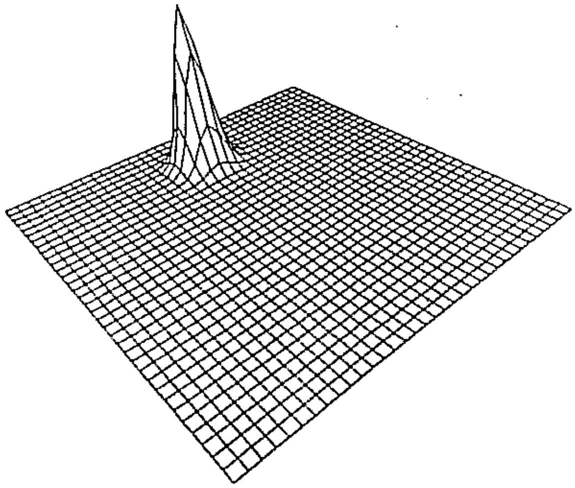

Figure 10. Initial condition and analytical solution after one complete rotation for a rotating 2-D Gaussian plume in pure convection ($\sigma = 264$)

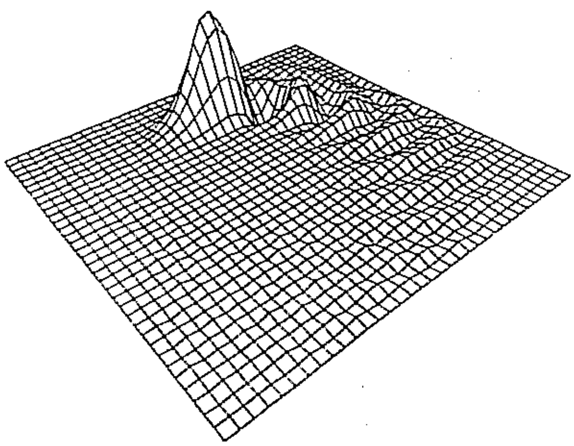

Figure 11. Case 3–1: Standard Bubnov–Galerkin solution of a rotating 2-D Gaussian plume after one complete rotation ($\sigma = 264$ and Courant number at the peak is 0·70)

criteria values are again summarized in Table II. Case 4–1 shown in Figure 14 indicates that the standard Bubnov–Galerkin solution for this wider and therefore easier problem is still quite poor, particularly the oscillation which follows the distribution is relatively large. The use of $N + 2$/TUSC upwinding, Case 4–3 shown in Figure 15, leads to an almost perfect solution. Peak and dip errors are in fact less than 0·5 per cent. The $\bar{\beta}_x$ and $\bar{\beta}_y$ coefficients were computed using $\lambda/h = 4{\cdot}0$, although using somewhat higher λ/h values leads to a slightly better solution.

The behaviour of the new $N + 2$/TUSC Petrov–Galerkin scheme with physical diffusion has been tested by considering a simplified form of equation (1) with no physical cross diffusion terms.

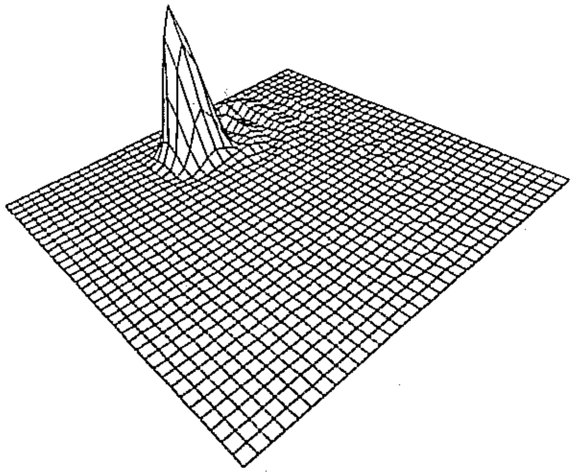

Figure 12. Case 3–3: $N + 2$/TUSC Petrov–Galerkin solution of a rotating 2-D Gaussian plume after one complete rotation ($\sigma = 264$ and Courant number at the peak is 0·70). ($\hat{\beta}_x$ and $\hat{\beta}_y$ are calculated from equation (18) with $\lambda/h = 4{\cdot}0$, $\hat{\beta}_{xy} = C_x C_y$)

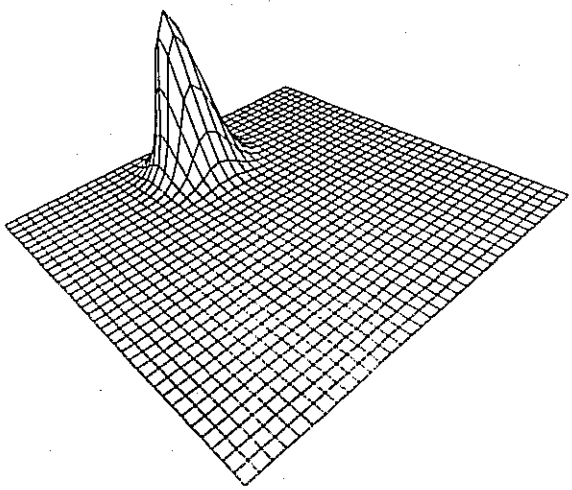

Figure 13. Initial condition and analytical solution after one complete rotation for a rotating 2-D Gaussian plume in pure convection ($\sigma = 400$)

The improvement in the time/space discretization characteristics of the new scheme for the convective part of the equation together with the physical diffusion leads to almost perfect solutions. The optimal upwinding coefficients appear quite insensitive to Peclet number for convection dominated flows, and a near optimum solution is always achieved for $\hat{\beta}_x$ and $\hat{\beta}_y$ corresponding to $\lambda/h = 4$ and $\hat{\beta}_{xy}$ specified using the truncation analysis prediction.

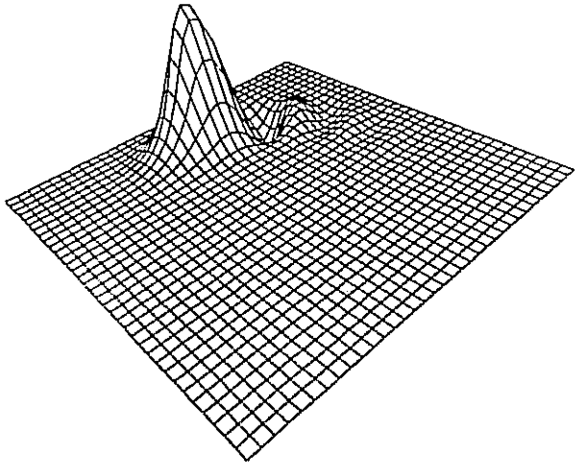

Figure 14. Case 4–1: Standard Bubnov–Galerkin solution of a rotating 2-D Gaussian plume after one complete rotation ($\sigma = 400$ and Courant number at the peak is 0·70)

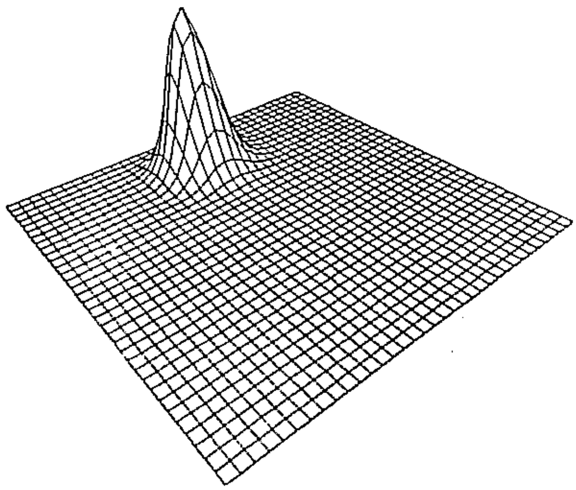

Figure 15. Case 4–3: $N + 2$/TUSC Petrov–Galerkin solution of a rotating 2-D Gaussian plume after one complete rotation ($\sigma = 400$ and Courant number at the peak is 0·70). ($\hat{\beta}_x$ and $\hat{\beta}_y$ are calculated from equation (18) with $\lambda/h = 4{\cdot}0$, $\hat{\beta}_{xy} = C_x C_y$)

CONCLUSIONS

We have successfully developed a higher order upwinded finite element scheme for convection dominated transport problems. In general, the $N + 2$/TUSC upwinding scheme leads to very accurate solutions owing to the excellent phase properties and perfect analytical damping of the scheme. The scheme is robust (i.e. $\hat{\beta}_x$, $\hat{\beta}_y$ and $\hat{\beta}_{xy}$ are simply defined and a function of essentially local Courant number only), very accurate and simple. The two cubic terms with coefficients $\hat{\beta}_x$ and $\hat{\beta}_y$ are one-dimensional terms and the third term is a sixth order cross term which enables very effective control in eliminating the cross-derivative truncation terms for general two-dimensional flows. It has to be noted that other terms in the general biasing function (12) do not allow this type of control. For this reason, straightforward extensions of upwinded Petrov–Galerkin schemes into two dimensions by simply taking the product of two one-dimensional terms (i.e. $N + 2$/SPE) have been unsatisfactory at high Courant numbers.

The $N + 2$/TUSC uni-directional upwinding coefficients are well predicted as a function of Courant number and λ/h using equation (18), which is based on Fourier analysis. The optimal $\hat{\beta}_x$ and $\hat{\beta}_y$ appear only weakly dependent on the wave number content of the distribution. Thus the value $\lambda/h = 4$ leads to a near optimal solution for a wide variety of flow fields, distribution sizes and Peclet numbers. Therefore for all practical purposes, $\hat{\beta}_x$ and $\hat{\beta}_y$ can be universally defined as being only Courant number dependent. The cross upwinding coefficient, $\hat{\beta}_{xy}$, is faithfully predicted by truncation error analysis and is solely Courant number dependent.

The $N + 2$/TUSC scheme leads to improved solutions over the entire Courant number range up to $C = 1{\cdot}0$. The effectiveness of the $N + 2$/TUSC scheme increases as Courant number increases. Nonetheless, the example problems indicate that, even at low Courant numbers such as $C = 0{\cdot}24$, substantial improvement over the standard Bubnov–Galerkin solution results when $N + 2$/TUSC upwinding is applied. Higher Courant number cases result in even better $N + 2$/TUSC solutions. Thus the $N + 2$/TUSC scheme's convergence curve with respect to time integration (i.e. error vs. C) has a negative slope (as opposed to standard Bubnov–Galerkin which has a positive slope) and always lies below the convergence curve for the standard Bubnov–Galerkin method.

It can be concluded that the newly developed $N + 2$/TUSC upwinded Petrov–Galerkin scheme is capable of handling convection dominated transient transport problems with considerable accuracy. The method enhances the solutions for a broad range of Courant numbers without requiring excessive programing or computational effort. Finally, the scheme is quite economical in that simple bilinear elements are used and in that relatively large time steps (with $C \simeq O(1)$) are allowed.

ACKNOWLEDGEMENTS

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APPENDIX

Derivation of the optimal form of the $N + 2$ degree biasing function in two dimensions through truncation error analysis

The nodal difference equation expanded in Taylor series and reorganized in terms of equal order spatial derivatives of ϕ is used to analyse the truncation error in the proposed scheme in order to assess the optimal values of the variable coefficients in the weighting function which is given in its general form in equation (13).

The first five terms of the truncation error are evaluated as

$$\tau_{i1} = \{0\}\phi + \{0\}\frac{\partial\phi}{\partial x} + \{0\}\frac{\partial\phi}{\partial y} + \left\{ - u\Delta h^2 k\left(\frac{30}{225}(a_{1,14} + a_{2,14} + a_{3,14} + a_{4,14}) + \frac{2}{30}(a_{1,7} + a_{2,7} + a_{3,7} + a_{4,7})\right)\right\}\frac{\partial^2\phi}{\partial x^2}$$

$$+\left\{-v\Delta hk^2\left(\frac{30}{225}(a_{1,15}+a_{2,15}+a_{3,15}+a_{4,15})\right.\right.$$
$$\left.\left.+\frac{2}{30}(a_{1,10}+a_{2,10}+a_{3,10}+a_{4,10})\right)\right\}\frac{\partial^2\phi}{\partial y^2} \tag{A1}$$

where h and k are the node to node distances in the x and y directions respectively.

It is obvious from equation (A1) that all truncation error terms associated with second order spatial derivatives can be easily eliminated by setting

$$\sum_{i=1}^{4} a_{i,7}=\sum_{i=1}^{4} a_{i,10}=\sum_{i=1}^{4} a_{i,14}=\sum_{i=1}^{4} a_{i,15}=0 \tag{A2}$$

With the conditions given in (A2) the remaining terms in the truncation error can be expressed as

$$\tau_{12}=\left\{u^3\frac{\Delta^3}{12}hk-u\Delta h^3k\left(\frac{15}{225}(a_{1,14}-a_{2,14}-a_{3,14}+a_{4,14})\right.\right.$$
$$\left.\left.+\frac{2}{60}(a_{1,7}-a_{2,7}-a_{3,7}+a_{4,7})\right)\right\}\frac{\partial^3\phi}{\partial x^3}$$
$$+\left\{v^3\frac{\Delta^3}{12}hk-v\Delta hk^3\left(\frac{15}{225}(a_{1,15}+a_{2,15}-a_{3,15}-a_{4,15})\right.\right.$$
$$\left.\left.+\frac{2}{60}(a_{1,10}+a_{2,10}-a_{3,10}-a_{4,10})\right)\right\}\frac{\partial^3\phi}{\partial y^3}$$
$$+\left\{u^2v\frac{\Delta^3}{4}hk-u\Delta h^2k^2\left(\frac{4}{225}(a_{1,16}+a_{2,16}+a_{3,16}+a_{4,16})\right.\right.$$
$$+\frac{1}{90}(a_{1,11}+a_{2,11}+a_{3,11}+a_{4,11})+\frac{1}{90}(a_{1,13}+a_{2,13}+a_{3,13}+a_{4,13})$$
$$\left.\left.+\frac{15}{225}(a_{1,14}+a_{2,14}-a_{3,14}-a_{4,14})+\frac{1}{30}(a_{1,7}+a_{2,7}-a_{3,7}-a_{4,7})\right)\right\}\frac{\partial^3\phi}{\partial x^2\partial y}$$
$$+\left\{uv^2\frac{\Delta^3}{4}hk-v\Delta h^2k^2\left(\frac{4}{225}(a_{1,16}+a_{2,16}+a_{3,16}+a_{4,16})\right.\right.$$
$$+\frac{1}{90}(a_{1,11}+a_{2,11}+a_{3,11}+a_{4,11})+\frac{1}{90}(a_{1,13}+a_{2,13}+a_{3,13}+a_{4,13})$$
$$\left.\left.+\frac{15}{225}(a_{1,15}-a_{2,15}-a_{3,15}+a_{4,15})+\frac{1}{30}(a_{1,10}-a_{2,10}-a_{3,10}+a_{4,10})\right)\right\}\frac{\partial^3\phi}{\partial x\partial y^2}$$
$$+\left\{-u^4\frac{\Delta^4}{24}hk+u^2\frac{\Delta^2}{2}h^3k\left(\frac{15}{225}(a_{1,14}-a_{2,14}-a_{3,14}+a_{4,14})\right.\right.$$
$$\left.\left.+\frac{2}{60}(a_{1,7}-a_{2,7}-a_{3,7}+a_{4,7})\right)\right\}\frac{\partial^4\phi}{\partial x^4}$$
$$+\left\{-v^4\frac{\Delta^4}{24}hk+v^2\frac{\Delta^2}{2}hk^3\left(\frac{15}{225}(a_{1,15}+a_{2,15}-a_{3,15}-a_{4,15})\right.\right.$$

$$+ \frac{2}{60}(a_{1,10} + a_{2,10} - a_{3,10} - a_{4,10})\Big)\Big\} \frac{\partial^4 \phi}{\partial y^4}$$

$$+ \Big\{ - uv^3 \frac{\Delta^4}{6} hk - v\Delta h^2 k^3 \Big(\frac{1}{30}(a_{1,15} - a_{2,15} + a_{3,15} - a_{4,15})$$

$$+ \frac{1}{60}(a_{1,10} - a_{2,10} + a_{3,10} - a_{4,10}) + \frac{4}{550}(a_{1,16} + a_{2,16} - a_{3,16} - a_{4,16})$$

$$+ \frac{1}{180}(a_{1,11} + a_{2,11} - a_{3,11} - a_{4,11}) + \frac{1}{180}(a_{1,13} + a_{2,13} - a_{3,13} - a_{4,13})\Big)$$

$$+ v^2 \Delta^2 h^2 k^2 \Big(\frac{1}{30}(a_{1,15} - a_{2,15} - a_{3,15} + a_{4,15}) + \frac{1}{60}(a_{1,10} - a_{2,10} - a_{3,10} + a_{4,10})$$

$$+ \frac{4}{550}(a_{1,16} + a_{2,16} + a_{3,16} + a_{4,16}) + \frac{1}{180}(a_{1,11} + a_{2,11} + a_{3,11} + a_{4,11})$$

$$+ \frac{1}{180}(a_{1,13} + a_{2,13} + a_{3,13} + a_{4,13})\Big)$$

$$+ uv\Delta^2 hk^3 \Big(\frac{1}{30}(a_{1,15} + a_{2,15} - a_{3,15} - a_{4,15})$$

$$+ \frac{1}{60}(a_{1,10} + a_{2,10} - a_{3,10} - a_{4,10})\Big)\Big\} \frac{\partial^4 \phi}{\partial x \partial y^3}$$

$$+ \Big\{ - u^3 v \frac{\Delta^4}{6} hk - u\Delta h^3 k^2 \Big(\frac{1}{30}(a_{1,14} - a_{2,14} + a_{3,14} - a_{4,14})$$

$$+ \frac{1}{60}(a_{1,7} - a_{2,7} + a_{3,7} - a_{4,7}) + \frac{4}{550}(a_{1,16} - a_{2,16} - a_{3,16} + a_{4,16})$$

$$+ \frac{1}{180}(a_{1,11} - a_{2,11} - a_{3,11} + a_{4,11}) + \frac{1}{180}(a_{1,13} - a_{2,13} - a_{3,13} + a_{4,13})\Big)$$

$$+ u^2 \Delta^2 h^2 k^2 \Big(\frac{1}{30}(a_{1,14} + a_{2,14} - a_{3,14} - a_{4,14}) + \frac{1}{60}(a_{1,7} + a_{2,7} - a_{3,7} - a_{4,7})$$

$$+ \frac{4}{550}(a_{1,16} + a_{2,16} + a_{3,16} + a_{4,16}) + \frac{1}{180}(a_{1,11} + a_{2,11} + a_{3,11} + a_{4,11})$$

$$+ \frac{1}{180}(a_{1,13} + a_{2,13} + a_{3,13} + a_{4,13})\Big)$$

$$+ uv\Delta^2 h^3 k \Big(\frac{1}{30}(a_{1,14} - a_{2,14} - a_{3,14} + a_{4,14}) + \frac{1}{60}(a_{1,7} - a_{2,7} - a_{3,7} + a_{4,7})\Big)\Big\} \frac{\partial^4 \phi}{\partial x^3 \partial y}$$

$$+ \Big\{ - u^2 v^2 \frac{\Delta^4}{4} hk - u\Delta h^2 k^3 \Big(\frac{4}{550}(a_{1,16} + a_{2,16} - a_{3,16} - a_{4,16})$$

$$+ \frac{1}{180}(a_{1,11} + a_{2,11} - a_{3,11} - a_{4,11}) + \frac{1}{180}(a_{1,13} + a_{2,13} - a_{3,13} - a_{4,13})\Big)$$

$$- v\Delta h^3 k^2 \left(\frac{4}{550}(a_{1,16} - a_{2,16} - a_{3,16} + a_{4,16}) + \frac{1}{180}(a_{1,11} - a_{2,11} - a_{3,11} + a_{4,11}) \right)$$

$$+ uv\Delta^2 h^2 k^2 \left(\frac{1}{30}(a_{1,14} + a_{2,14} - a_{3,14} - a_{4,14}) + \frac{1}{60}(a_{1,7} + a_{2,7} - a_{3,7} - a_{4,7}) \right.$$

$$+ \frac{1}{30}(a_{1,15} - a_{2,15} - a_{3,15} + a_{4,15}) + \frac{1}{60}(a_{1,10} - a_{2,10} - a_{3,10} + a_{4,10})$$

$$+ \frac{4}{225}(a_{1,16} + a_{2,16} + a_{3,16} + a_{4,16}) + \frac{1}{90}(a_{1,11} + a_{2,11} + a_{3,11} + a_{4,11})$$

$$\left. \left. + \frac{1}{90}(a_{1,13} + a_{2,13} + a_{3,13} + a_{4,13}) \right) \right\} \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \cdots + H.O.T. \tag{A3}$$

At this point we will assume that the truncation error associated with the third order derivatives in x and y directions will be eliminated by solely using the uni-directional $a_{i,7}$ and $a_{i,10}$ terms. This assures compatibility between the one- and two-dimensional schemes. Therefore, from the first two terms of the truncation error expression we get the following conditions:

$$(a_{1,7} - a_{2,7} - a_{3,7} + a_{4,7}) = \frac{5}{2}\frac{u^2\Delta^2}{h^2} \tag{A4}$$

$$(a_{1,14} - a_{2,14} - a_{3,14} + a_{4,14}) = 0 \tag{A5}$$

$$(a_{1,10} + a_{2,10} - a_{3,10} - a_{4,10}) = \frac{5}{2}\frac{v^2\Delta^2}{k^2} \tag{A6}$$

$$(a_{1,15} + a_{2,15} - a_{3,15} - a_{4,15}) = 0 \tag{A7}$$

In order to eliminate the truncation error associated with the third order cross derivatives we must have

$$\frac{4}{225}\sum_{i=1}^{4} a_{i,16} + \frac{1}{90}\sum_{i=1}^{4} a_{i,13} + \frac{1}{90}\sum_{i=1}^{4} a_{i,11} = \frac{uv\Delta^2}{4hk} \tag{A8}$$

It has to be noted that, since $a_{i,14}$ and $a_{i,7}$ are functions of u^2, and $a_{i,15}$ and $a_{i,10}$ are functions of v^2 only, we do not expect them to contribute to the elimination of these particular truncation error terms. Therefore we set them all equal to zero.

$$(a_{1,14} + a_{2,14} - a_{3,14} - a_{4,14}) = 0 \tag{A9}$$

$$(a_{1,7} + a_{2,7} - a_{3,7} - a_{4,7}) = 0 \tag{A10}$$

$$(a_{1,15} - a_{2,15} - a_{3,15} + a_{4,15}) = 0 \tag{A11}$$

$$(a_{1,10} - a_{2,10} - a_{3,10} + a_{4,10}) = 0 \tag{A12}$$

The conditions given in equations (A2) and (A4) to (A12) eliminate all the truncation errors up to fourth order derivatives. Following this procedure and after the subsequent substitution of the equations into the remaining terms of truncation error given in equation (A3), it is possible to eliminate all the truncation error terms up to fifth order derivatives provided that the following conditions are satisfied.

From $\partial^4\phi/\partial x^3\partial y$ terms:

$$(a_{1,15} - a_{2,15} + a_{3,15} - a_{4,15}) = 0 \tag{A13}$$

$$(a_{1,10} - a_{2,10} + a_{3,10} - a_{4,10}) = 0 \tag{A14}$$

$$\tfrac{4}{550}(a_{1,16} + a_{2,16} - a_{3,16} - a_{4,16}) + \tfrac{1}{180}(a_{1,11} + a_{2,11} - a_{3,11} - a_{4,11}) + \tfrac{1}{180}(a_{1,13} + a_{2,13} - a_{3,13} - a_{4,13}) = 0 \tag{A15}$$

and from $\partial^4\phi/\partial x\partial y^3$ terms:

$$(a_{1,14} - a_{2,14} + a_{3,14} - a_{4,14}) = 0 \tag{A16}$$

$$(a_{1,7} - a_{2,7} + a_{3,7} - a_{4,7}) = 0 \tag{A17}$$

$$\tfrac{4}{225}(a_{1,16} - a_{2,16} - a_{3,16} + a_{4,16}) + \tfrac{1}{90}(a_{1,11} - a_{2,11} - a_{3,11} + a_{4,11}) + \tfrac{1}{90}(a_{1,13} - a_{2,13} - a_{3,13} + a_{4,13}) = 0 \tag{A18}$$

Solving equations (A4) to (A18) simultaneously together with (A2) we get

$$a_{1,7} = -a_{2,7} = -a_{3,7} = a_{4,7} = \tfrac{5}{8}C_x^2 \tag{A19}$$

$$a_{1,10} = a_{2,10} = -a_{3,10} = -a_{4,10} = \tfrac{5}{8}C_y^2 \tag{A20}$$

$$a_{1,14} = a_{2,14} = a_{3,14} = a_{4,14} = 0 \tag{A21}$$

$$a_{1,15} = a_{2,15} = a_{3,15} = a_{4,15} = 0 \tag{A22}$$

where $C_x = u\Delta/h$ and $C_y = v\Delta/k$ are the elemental Courant numbers in the x and y directions respectively.

A major problem is encountered in evaluating the values of the coefficients $a_{i,11}$, $a_{i,13}$ and $a_{i,16}$ in that we lack an adequate number of constraints in order to determine these coefficients algebraically. Therefore, through a series of numerical experiments the relative contribution of each of these terms was assessed. These experiments showed that when terms from equation (15) with non-zero $a_{i,11}$ and $a_{i,13}$ values are included, the resulting scheme does not improve the solution. With this in mind, we decided to consider only the $a_{i,16}$ terms which represent the complete sixth degree cross terms in our generalized weighting function. At this point we assume that all $a_{i,16}$ values are equal to each other and hence from equation (A8) we get

$$a_{1,16} = a_{2,16} = a_{3,16} = a_{4,16} = \tfrac{225}{64}C_xC_y \tag{A23}$$

Using equations (15), (A19), (A20) and (A23) leads to the weighting functions for $N + 2$ upwinded Petrov–Galerkin type finite element schemes in two dimensions, as given in equation (17).

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