LECTURE 10

APPLICATIONS OF FD APPROXIMATIONS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

Ordinary Differential Equations

Initial Value Problems

- For Initial Value problems (IVP's), conditions are specified at only one value of the independent variable $\rightarrow$ initial conditions (i.c.'s)

- For example a simple harmonic oscillator is described by

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = g(t) \qquad y(0) = y_o \qquad \frac{dy}{dt}(0) = V_o$$

  - y = location $\rightarrow$ dependent variable
  - t = time $\rightarrow$ independent variable

Boundary Value Problems

- For Boundary Value Problems (BVP's) conditions are specified at two values of the independent variable (which represent the actual physical boundaries)
- Example

$$\frac{d^2y}{dx^2} + D\frac{dy}{dx} + Ey = h(x) \qquad y(0) = y_o \qquad y(L) = y_l$$

General Initial Value Problems

- Any IVP can be represented as a set of one or more 1st order d.e.'s each with an i.c.
- Example

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + C = g(t) \qquad y(0) = y_o \qquad \frac{dy}{dt}(0) = v_o$$

- Let $z = \frac{dy}{dt}$ and we can develop a system of 2 first order O.D.E.'s which are coupled

$$\frac{dy}{dt} = z \qquad y(0) = y_o$$

$$\frac{dz}{dt} = -\frac{B}{A}z - \frac{C}{A} + \frac{g(t)}{A} \qquad z(0) = v_o$$

- Therefore the general IVP can be written as:

$$\frac{dy_1}{dt} = f_1(y_1, y_2, \ldots, y_n, t) \qquad y_1(0) = y_{10}$$

$$\vdots$$

$$\frac{dy_n}{dt} = f_n(y_1, y_2, \ldots, y_n, t) \qquad y_n(0) = y_{n0}$$

- Possible solution strategies for higher order o.d.e.'s
  - Solutions of 1st order d.e.'s
    - single equation with associated i.c.
    - extension to coupled sets
  - Solution of higher order equations without reduction to a 1st order system → must develop a method for a specific equation
    - not as general
    - may be advantageous in certain cases

Solution to a 1st Order Single Equation IVP

$$\frac{dy}{dt} = f(y, t) \quad \text{with specified i.c.} \quad y(t_o) = y_o$$

Euler Method

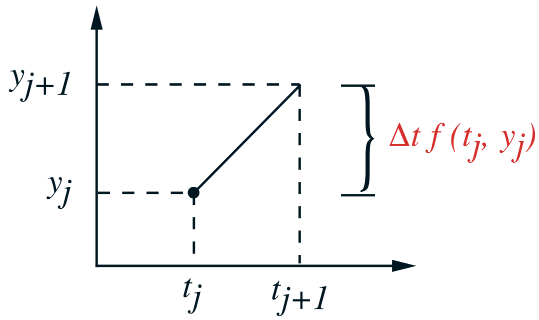
- The Euler method is a 1st order method
- We evaluate the o.d.e. at node j and use a forward difference approximation for $\left.\frac{dy}{dt}\right|_j$

$$\left.\frac{dy}{dt}\right|_j = f(y_j, t_j) \qquad \Rightarrow$$

$$\frac{y_{j+1} - y_j}{\Delta t} = f(y_j, t_j) \qquad \Rightarrow$$

$$\mathbf{y_{j+1} = y_j + \Delta t\, f(y_j, t_j)}$$

- ***Simply "march" forward in time***
    - From time level j (time = t_j)
    - To time level $j+1$ (time = $t_{j+1} = t_j + \Delta t$)

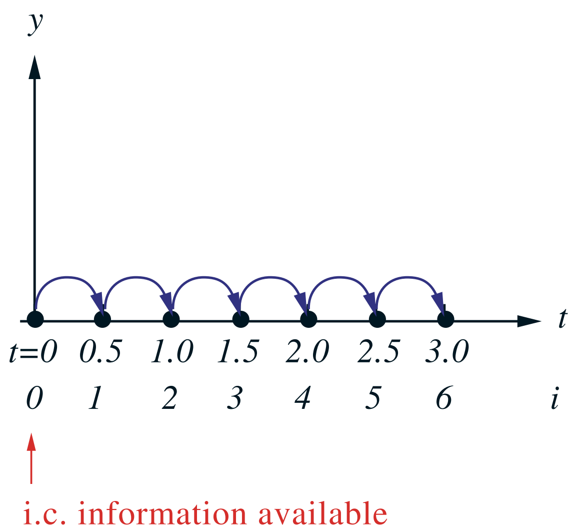


- We note that $f(t_j, y_j)$ equals the slope at t_j.
- Therefore to obtain y_{j+1} simply add $\Delta t\ f(t_j, y_j)$ to y_j

Example

- Solve $\frac{dy}{dt} = -y|y|$ with the i.c. $y(0) = 1$

- Apply the Euler approximation to solve this IVP equation. Apply a time step equal to $\Delta t = 0.5$. Solve up to $t = 3$ $(0 \le t \le 3)$



- Discretize the o.d.e. at a general node i

$$\left.\frac{dy}{dt}\right|_i = -y_i|y_i|$$

- Approximate $\left.\frac{dy}{dt}\right|_i$ using a forward difference approximation

$$\frac{y_{i+1} - y_i}{\Delta t} = -y_i|y_i|$$

$$\Rightarrow$$

$$y_{i+1} = y_i - \Delta t \, y_i|y_i|$$

Next Value = Previous Value + Run $\times$ Slope

- Equation relates a known time level i to the new time level $i+1$. ***This process is known as "time stepping" or "time marching"***

- The i.c. indicates that $y_o = 1$

- Take 1st time step $\Rightarrow \quad i = 0 \quad \rightarrow \quad i+1 = 1$

    - Note that $t_i = i \cdot \Delta t + t_o = i\Delta t$ where $t_o = 0$ in this case

$$y_1 = y_o - \Delta t y_o |y_o|$$

$$y_1 = 1 - 0.5 \times 1 \times |1|$$

$$y_1 = 0.5 \quad \text{at} \quad t_1 = 0.5$$

- Take next time step $i = 1 \quad \rightarrow \quad i+1 = 2$

$$y_2 = y_1 - \Delta t \; y_1 |y_1|$$

$$y_2 = 0.5 - 0.5 \times 0.5 \times |0.5|$$

$$y_2 = 0.375 \quad \text{at} \quad t_2 = 1.0$$

- Take next time step $i = 2 \rightarrow i+1 = 3$

$$y_3 = y_2 - \Delta t\, y_2 |y_2|$$

$$y_3 = 0.375 - 0.5 \times (0.375) \times |0.375|$$

$$y_3 = 0.30469 \quad \text{at} \quad t_3 = 1.5$$

- Take next time step $i = 3 \rightarrow i+1 = 4$

$$y_4 = y_3 - \Delta t\, y_3 |y_3|$$

$$y_4 = 0.30469 - 0.5 \times (0.30469) \times |0.30469|$$

$$y_4 = 0.25827 \quad \text{at} \quad t_4 = 2.0$$

- Take next time step

$$y_5 = 0.22492 \quad \text{at} \quad t_5 = 2.5$$

- Take next time step

$$y_6 = 0.19963 \quad \text{at} \quad t_6 = 3.0$$

- We can continue time marching

t	Numerical Solution using $\Delta t = 0.5$ and the Euler Method	Numerical Solution using $\Delta t = 0.1$ and the Euler Method	Exact Solution
0.0	1.0000 (i.c.)	1.0000 (i.c.)	1.0000
1.0	0.37500	0.4982	0.5000
2.0	0.25827	0.3321	0.3333
3.0	0.19963	0.2491	0.2580

General Observations for Solving IVP's

- Solution to o.d.e.'s can be very simple using finite difference approximations to represent differentiation
- Accuracy is dependent on the time step Δt! We need to understand the error behavior
- As $\Delta t \rightarrow 0$, the solution gets better
- IVP's are solved using a time marching process $\rightarrow$ Begin at one end and march forward up to the desired point or indefinitely
- At each time step, we introduce a new unknown, y_{j+1}, which is solved for by writing and solving the discrete form of the IVP at node *j*.

Solutions to Boundary Value Problems

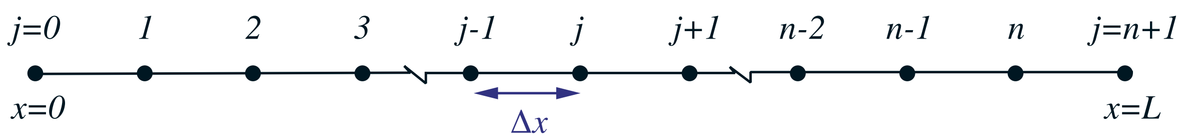
- Boundary value problems must be 2nd order o.d.e.'s or higher
- We apply FD approximations to the various terms in the differential equation to obtain discrete approximations to the differential equations at points in space.
- Unknown functional values at the nodes will be coupled and require the solution of a system of simultaneous equations $\rightarrow$ matrix methods.

Example

- Consider a steady state 1-D problem $\frac{d^2y}{dx^2} + Ay = B$

  with b.c.'s specified as $y(0) = 0$ and $y(L) = 0$

- Consider the following discretization of the domain



- ***At each node, we introduce an unknown y_j (total of n+2 unknowns)***

- The O.D.E. is approximated at generic node j as

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{(\Delta x)^2} + Ay_j = B$$

- We have used a central difference approximation of $O(\Delta x)^2$

- Therefore we generate $n+2$ equations (1 for each interior point and 1 for each boundary node) to solve for the $n+2$ unknowns:

$$y_0 = 0$$

$$y_2 - 2y_1 + y_0 + A(\Delta x)^2 y_1 = B(\Delta x)^2$$

$$y_3 - 2y_2 + y_1 + A(\Delta x)^2 y_2 = B(\Delta x)^2$$

$$\vdots$$

$$y_{j+1} - 2y_j + y_{j-1} + A(\Delta x)^2 y_j = B(\Delta x)^2$$

$$\vdots$$

$$y_n - 2y_{n-1} + y_{n-2} + A(\Delta x)^2 y_{n-1} = B(\Delta x)^2$$

$$y_{n+1} - 2y_n + y_{n-1} + A(\Delta x)^2 y_n = B(\Delta x)^2$$

$$y_{n+1} = 0$$

- Collect coefficients of unknowns and write in matrix form:

$$\begin{bmatrix} 1 & 0 & & & & & & & & & & \\ 1 & \alpha & 1 & & & & & & & & & \\ & 1 & \alpha & 1 & & & & & & & & \\ & & 1 & \alpha & 1 & & & & & & & \\ & & & . & . & . & & & & & & \\ & & & & . & . & . & & & & & \\ & & & & & . & . & . & & & & \\ & & & & & & 1 & \alpha & 1 & & & \\ & & & & & & & 1 & \alpha & 1 & & \\ & & & & & & & & 1 & \alpha & 1 \\ & & & & & & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \\ \\ \\ y_{n-1} \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ B(\Delta x)^2 \\ B(\Delta x)^2 \\ B(\Delta x)^2 \\ \\ \\ \\ B(\Delta x)^2 \\ B(\Delta x)^2 \\ 0 \end{bmatrix}$$

where

$$\alpha = -2 + A(\Delta x)^2$$

- Notes on solutions
  - Tri-diagonal systems are very inexpensive to solve when a specialized compact storage tri-diagonal solver is used
  - We can reduce the system of simultaneous equations from $(n+2)\times(n+2)$ to $n\times n$ by incorporating the b.c.'s into the discrete form of the differential equation at nodes 1 and n
  - If the o.d.e. is nonlinear $\rightarrow$ the method no longer generates linear set of simultaneous algebraic equations but a ***nonlinear*** set!

Example

- Solve the following nonlinear o.d.e.

$$\frac{d^2y}{dx^2} + Dy^2 = E$$

- Using a central difference approximation yields nonlinear algebraic equations

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{(\Delta x)^2} + Dy_j^2 = E$$

- Solution strategies include:
  - Iterative solution of algebraic equations puts the nonlinear term on the r.h.s. and iterates until convergence. There may be convergence problems.
  - Linearization of the nonlinear terms. Use Taylor series to approximate the nonlinear terms.