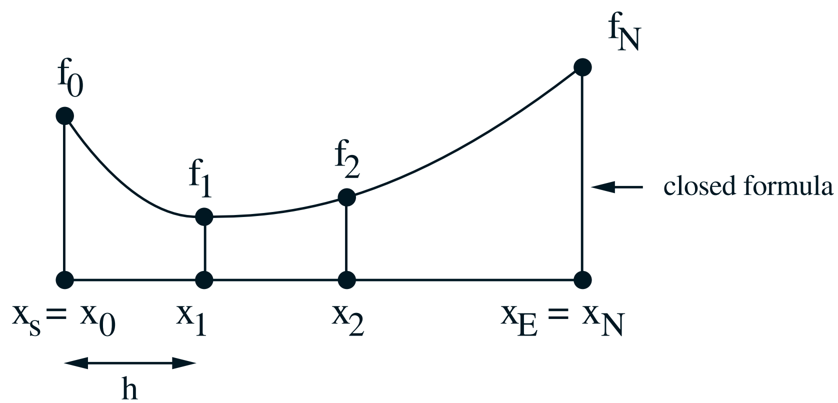
LECTURE 16

GAUSS QUADRATURE

- In general for Newton-Cotes (equispaced interpolation points/ data points/ integration points/ nodes).

$$\int_{x_S}^{x_E} f(x)dx = h[w_o' f_o + w_1' f_1 + \ldots + w_N' f_N] + E$$



- Note that for Newton-Cotes formulae only the weighting coefficients w_i were unknown and the x_i were fixed

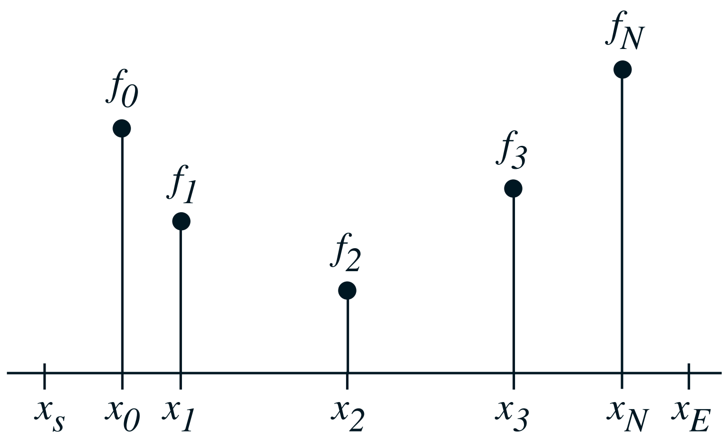
- However the number of and placement of the integration points influences the accuracy of the Newton-Cotes formulae:
  - N even $\rightarrow N^{th}$ degree interpolation function exactly integrates an $N+1^{th}$ degree polynomial $\rightarrow$ This is due to the placement of one of the data points.
  - N odd $\rightarrow N^{th}$ degree interpolation function exactly integrates an N^{th} degree polynomial.

- ***Concept: Let's allow the placement of the integration points to vary such that we further increase the degree of the polynomial we can integrate exactly for a given number of integration points.***

- ***In fact we can integrate an $2N+1$ degree polynomial exactly with only $N+1$ integration points***

- Assume that for Gauss Quadrature the form of the integration rule is

$$\int_{x_S}^{x_E} f(x)dx = [w_o f_o + w_1 f_1 + \ldots + w_N f_N] + E$$



- In ***deriving*** (not applying) these integration formulae
  - Location of the integration points, x_i $i = O, N$ are unknown
  - Integration formulae weights, w_i $i = O, N$ are unknown
- $2(N+1)$ unknowns $\rightarrow$ we will be able to exactly integrate any $2N+1$ degree polynomial!

Derivation of Gauss Quadrature by Integrating Exact Polynomials and Matching

Derive 1 point Gauss-Quadrature

- 2 unknowns w_o, x_o which will exactly integrate any linear function

- Let the ***general*** polynomial be

$$f(x) = Ax + B$$

  where the coefficients A, B can equal any value

- Also consider the integration interval to be $[-1, +1]$ such that $x_S = -1$ and $x_E = +1$ (no loss in generality since we can always transform coordinates).

$$\int_{-1}^{+1} f(x)dx = w_o f(x_o)$$

- Substituting in the form of $f(x)$

$$\int_{-1}^{+1} (Ax + B)dx = w_o(Ax_o + B) \quad \Rightarrow$$

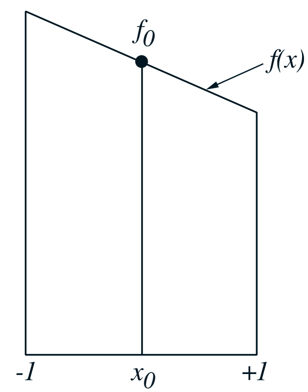
$$\left[A\frac{x^2}{2} + Bx\right]_{-1}^{+1} = w_o(Ax_o + B) \quad \Rightarrow$$

$$A(0) + B(2) = A(x_o w_o) + B(w_o)$$

- In order for this to be true for <u>any</u> 1st degree polynomial (i.e. any A and B).

$$\begin{cases} 0 = x_o w_o \\ 2 = w_o \end{cases}$$

- Therefore $\mathbf{x_o = 0}$, $\mathbf{w_o = 2}$ for 1 point $(N = 1)$ Gauss Quadrature.



- We can integrate exactly with only 1 point for a linear function while for Newton-Cotes we needed two points!

Derive a 2 point Gauss Quadrature Formula

- The general form of the integration formula is

$$I = w_o f_o + w_1 f_1$$

- w_o, x_o, w_1, x_1 are all unknowns

- 4 unknowns ⇒ we can fit a 3rd degree polynomial exactly

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

- Substituting in for $f(x)$ into the general form of the integration rule

$$\int_{-1}^{+1} f(x)dx = w_o f(x_o) + w_1 f(x_1)$$

$$\Rightarrow$$

$$\int_{-1}^{+1} [Ax^3 + Bx^2 + Cx + D]dx = w_o[Ax_o^3 + Bx_o^2 + Cx_o + D] + w_1[Ax_1^3 + Bx_1^2 + Cx_1 + D]$$

$$\Rightarrow$$

$$\left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx\right]_{-1}^{+1} = w_o(Ax_o^3 + Bx_o^2 + Cx_o + D) + w_1(Ax_1^3 + Bx_1^2 + Cx_1 + D)$$

$$\Rightarrow$$

$$A[w_o x_o^3 + w_1 x_1^3] + B\left[w_o x_o^2 + w_1 x_1^2 - \frac{2}{3}\right] + C[w_o x_o + w_1 x_1] + D[w_o + w_1 - 2] = 0$$

- In order for this to be true for ***any*** third degree polynomial (i.e. all arbitrary coefficients, A, B, C, D), we must have:

$$w_o x_o^3 + w_1 x_1^3 = 0$$

$$w_o x_o^2 + w_1 x_1^2 - \frac{2}{3} = 0$$

$$w_o x_o + w_1 x_1 = 0$$

$$w_o + w_1 - 2 = 0$$

- 4 nonlinear equations → 4 unknowns

$$w_o = 1 \quad \text{and} \quad w_1 = 1$$

$$x_o = -\sqrt{\frac{1}{3}} \quad \text{and} \quad x_1 = +\sqrt{\frac{1}{3}}$$

- All polynomials of degree 3 or less will be ***exactly*** integrated with a Gauss-Legendre 2 point formula.

Gauss Legendre Formulae

$$I = \int_{-1}^{+1} f(x)dx = \sum_{i=0}^{N} w_i f_i + E$$

N	$N+1$	x_i, $i = 0, N$	w_i	Exact for polynomials of degree
0	1	0	2	1
1	2	$-\sqrt{\frac{1}{3}}, +\sqrt{\frac{1}{3}}$	1, 1	3
2	3	-0.774597, 0, +0.774597	0.5555, 0.8889, 0.5555	5
N	$N+1$			$2N+1$

N	$N+1$	x_i, $i = 0, N$	w_i	Exact for polynomials of degree
3	4	-0.86113631 -0.33998104 0.33998104 0.86113631	0.34785485 0.65214515 0.65214515 0.34785485	7
4	5	-0.90617985 -0.53846931 0.00000000 0.53846931 0.90617985	0.23692689 0.47862867 0.56888889 0.47862867 0.23692689	9
5	6	-0.93246951 -0.66120939 -0.23861919 0.23861919 0.66120939 0.93246951	0.17132449 0.36076157 0.46791393 0.46791393 0.36076157 0.17132449	11

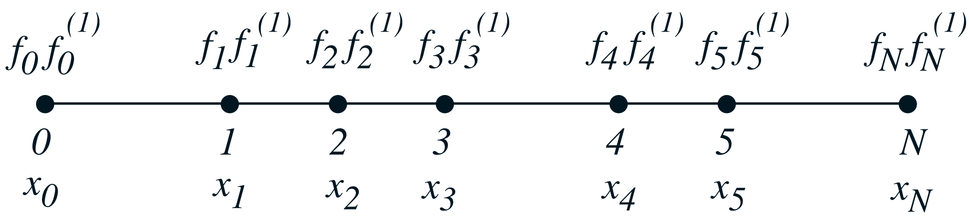
- Notes
  - $N+1$ = the number of integration points
  - Integration points are symmetrical on $[-1, +1]$
  - Formulae can be applied on any interval using a coordinate transformation
  - $N+1$ integration points $\rightarrow$ will integrate polynomials of up to degree $2N+1$ exactly.
    - Recall that Newton Cotes $\rightarrow$ $N+1$ integration points only integrates an $N^{th}/N+1^{th}$ degree polynomial exactly depending on N being odd or even.
    - For Gauss-Legendre integration, we allowed both weights and integration point locations to vary to match an integral exactly $\Rightarrow$ more d.o.f. $\Rightarrow$ allows you to match a higher degree polynomial!
    - An alternative way of looking at Gauss-Legendre integration formulae is that we use Hermite interpolation instead of Lagrange interpolation! (How can this be since Hermite interpolation involves derivatives $\rightarrow$ let's examine this!)

Derivation of Gauss Quadrature by Integrating Hermite Interpolating Functions

Hermite interpolation formulae

- Hermite interpolation which ***matches*** the function and the first derivative at $N+1$ interpolation points is expressed as:

$$g(x) = \sum_{i=0}^{N} \alpha_i(x) f_i + \sum_{i=0}^{N} \beta_i(x) f_i^{(1)}$$



- It can be shown that in general for non-equispaced points

$$\alpha_i(x) = t_i(x) l_{iN}(x) l_{iN}(x) \qquad i = 0, N$$

$$\beta_i(x) = s_i(x) l_{iN}(x) l_{iN}(x) \qquad i = 0, N$$

where

$$p_N(x) \equiv (x - x_o)(x - x_1)\cdots(x - x_N)$$

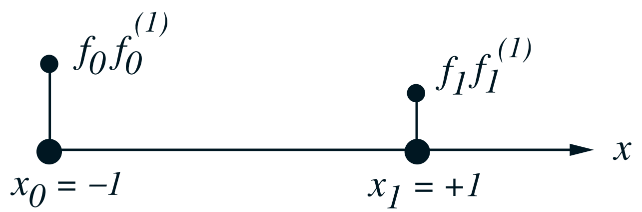
$$l_{iN}(x) \equiv \frac{p_N(x)}{(x - x_i)p_N^{(1)}(x_i)} \qquad i = 0, N$$

$$t_i(x) \equiv 1 - (x - x_i)\ 2\ l_{iN}^{(1)}(x_i)$$

$$s_i(x) \equiv (x - x_i)$$

Example of defining a cubic Hermite interpolating function

- Derive Hermite interpolating functions for 2 interpolation points located at -1 and $+1$ for the interval $[-1, +1]$.



$$N + 1 = 2 \quad \text{points} \quad \Rightarrow \quad N = 1$$

- Establish $p_N(x)$

$$p_1(x) = (x - x_o)(x - x_1) \quad \Rightarrow$$

$$p_1^{(1)}(x) = (x - x_o) + (x - x_1)$$

- Establish $l_{iN}(x)$

$$l_{i1}(x) = \frac{p_1(x)}{(x-x_i)[p_1^{(1)}(x_i)]}$$

$$l_{i1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_i)[(x_i-x_o)+(x_i-x_1)]}$$

- Let $i = 0$

$$l_{o1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_o)[0+(x_o-x_1)]} \quad \Rightarrow$$

$$l_{o1}(x) = \frac{x-x_1}{x_o-x_1}$$

- Substitute in $x_o = -1$ and $x_1 = +1$

$$l_{o1}(x) = \frac{1}{2}(1-x)$$

- Let $i = 1$

$$l_{11}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_1)[(x_1-x_o)+0]}$$

- Substitute in values for x_o, x_1

$$l_{11}(x) = \frac{1}{2}(1+x)$$

- Taking derivatives

$$l_{o1}^{(1)}(x) = -\frac{1}{2}$$

$$l_{11}^{(1)}(x) = +\frac{1}{2}$$

- Establish $t_i(x)$

$$t_o(x) = 1 - (x - x_o)2l_{o1}^{(1)}(x_o) \qquad \Rightarrow$$

$$t_o(x) = 1-(x+1)(2)\left(-\frac{1}{2}\right) \qquad \Rightarrow$$

$$t_o(x) = 2 + x$$

$$t_1(x) = 1 - (x - x_1)\ 2\ l_{11}^{(1)}(x_1) \qquad \Rightarrow$$

$$t_1(x) = 1 - (x - 1)\left(2 \cdot \frac{1}{2}\right) \qquad \Rightarrow$$

$$t_1(x) = 2 - x$$

- Establish $s_i(x)$

$$s_o(x) = x + 1$$

$$s_1(x) = x - 1$$

- Establish $\alpha_i(x)$

$$\alpha_o(x) = t_o(x) l_{o1}(x) l_{o1}(x) \quad \Rightarrow$$

$$\alpha_o(x) = (2+x)\frac{1}{2}(1-x)\frac{1}{2}(1-x) \quad \Rightarrow$$

$$\boldsymbol{\alpha_o(x) = \frac{1}{4}(2 - 3x + x^3)}$$

$$\alpha_1(x) = t_1(x) l_{11}(x) l_{11}(x) \quad \Rightarrow$$

$$\alpha_1(x) = (2-x)\frac{1}{2}(1+x)\frac{1}{2}(1+x) \quad \Rightarrow$$

$$\boldsymbol{\alpha_1(x) = \frac{1}{4}(2 + 3x - x^3)}$$

- Establish $\beta_i(x)$

$$\beta_o(x) = s_o(x) l_{o1}(x) l_{o1}(x) \quad \Rightarrow$$

$$\beta_o(x) = (x+1)\frac{1}{2}(1-x)\frac{1}{2}(1-x) \quad \Rightarrow$$

$$\boldsymbol{\beta_o(x) = \frac{1}{4}(1 - x - x^2 + x^3)}$$

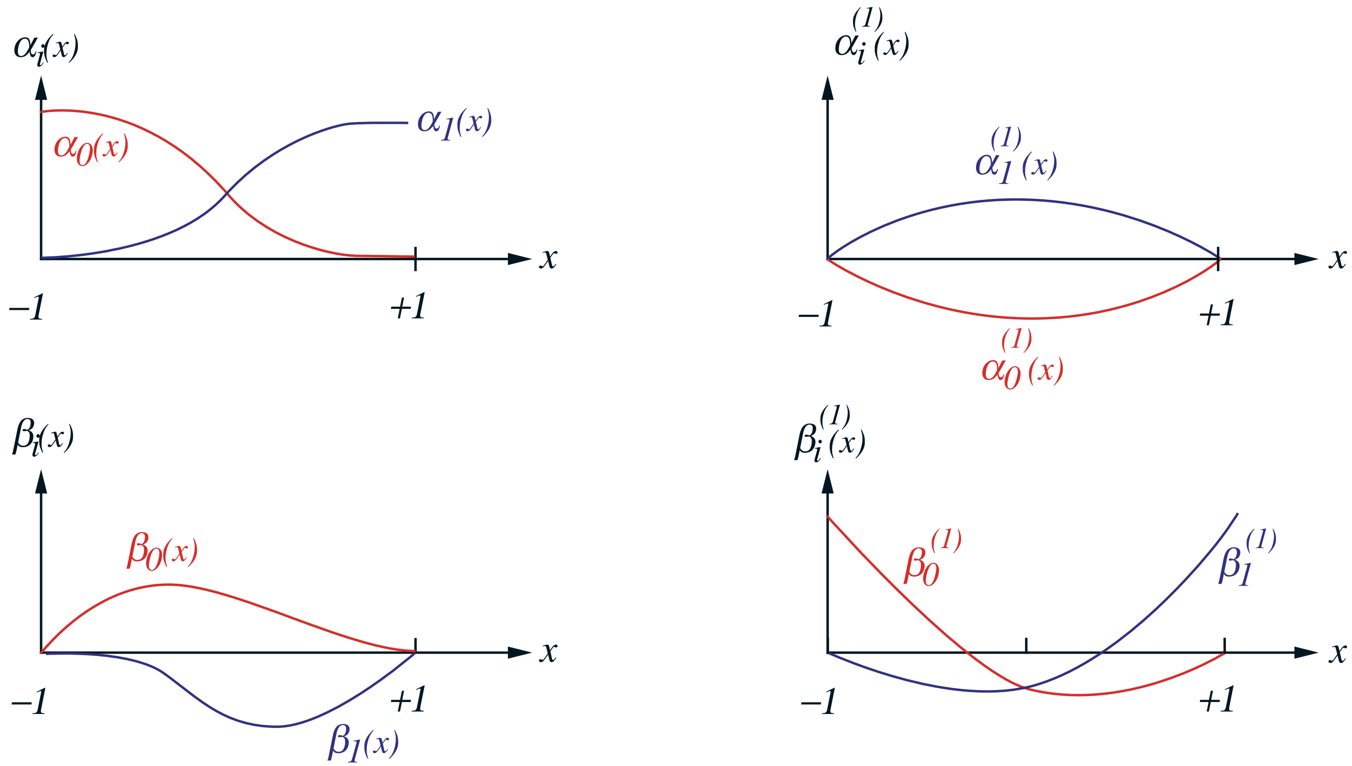
$$\beta_1(x) = s_1(x) l_{11}(x) l_{11}(x) \quad \Rightarrow$$

$$\beta_1(x) = (x-1)\frac{1}{2}(1+x)\frac{1}{2}(1+x) \quad \Rightarrow$$

$$\boldsymbol{\beta_1(x) = \frac{1}{4}(-1 - x + x^2 + x^3)}$$

- In general

$$\boldsymbol{g(x) = \alpha_o(x) f_o + \alpha_1(x) f_1 + \beta_o(x) f_o^{(1)} + \beta_1(x) f_1^{(1)}}$$



- These functions satisfy the constraints

$$\alpha_i(x_j) = \delta_{ij} \qquad \alpha_i^{(1)}(x_j) = 0$$

$$\beta_i(x_j) = 0 \qquad \beta_i^{(1)}(x_j) = \delta_{ij}$$

Gauss-Legendre Quadrature by integrating Hermite interpolating polynomials

$$I = \int_{-1}^{+1} f(x)dx = \sum_{i=0}^{N} w_i f_i + E$$

- Notes
  - Use $[-1, +1]$ without loss of generality $\Rightarrow$ we can always transform the interval.
  - Approximation for I is exact for $2N+1$ degree polynomials

- We can derive all Gauss-Legendre quadrature formulae by approximating $f(x)$ with an $2N+1^{th}$ degree Hermite interpolating function ***using*** N ***specially selected*** integration/interpolation points.

$$I = \int_{-1}^{+1} g(x)dx + E$$

where

$$g(x) = \sum_{i=0}^{N} \alpha_i(x) f_i + \sum_{i=0}^{N} \beta_i(x) f_i^{(1)}$$

- Thus

$$I = \int_{-1}^{+1} \left[\sum_{i=0}^{N} \alpha_i(x) f_i + \sum_{i=0}^{N} \beta_i(x) f_i^{(1)} \right] dx + E$$

$$\Rightarrow$$

$$I = \sum_{i=0}^{N} A_i f_i + \sum_{i=0}^{N} B_i f_i^{(1)} + E$$

where

$$A_i \equiv \int_{-1}^{+1} \alpha_i(x) dx \quad \text{and} \quad B_i \equiv \int_{-1}^{+1} \beta_i(x) dx$$

- Furthermore we can show that

$$E = \int_{-1}^{+1} \left[\frac{p_{N+1}^2(x)}{(2N+2)!} f^{(2N+2)}(x_o) + \text{H.O.T.} \right] dx$$

- Note that we are assuming Taylor series expansions about x_o and using higher order terms in the expansion.
  - Therefore $E = 0$ for any polynomial of degree $2N+1$ or less!
- The problem that we encounter is that the integration formula as it now stands ***in general*** requires us to know both functional and first derivative values at the nodes!
- Let us select $x_o, x_1, x_2, \ldots x_N$ such that

$$B_i = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_i(x)dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} s_i(x) l_{iN}(x) l_{iN}(x)dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\int_{-1}^{+1} (x-x_i)\frac{p_N(x)}{(x-x_i)p_N^{(1)}(x_i)}\, l_{iN}(x)dx = 0 \quad i = 0, N \quad \Rightarrow$$

$$\frac{1}{p_N^{(1)}(x_i)} \int_{-1}^{+1} p_N(x) l_{iN}(x) dx = 0 \qquad i = 0, N$$

$$p_N(x) \quad \Rightarrow \quad \text{polynomial of degree } N+1$$

$$l_{iN}(x) \quad \Rightarrow \quad \text{polynomial of degree } N$$

- Therefore we require $p_N(x)$ to be orthogonal on $[-1, +1]$ to ***all*** polynomials of degree N or less $\Rightarrow$ any multiple of Legendre-Polynomials will satisfy this.
- Let

$$p_N(x) = \frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!} P_{N+1}(x)$$

where

$$p_N(x) = (x - x_o)(x - x_1)(x - x_2)\dots(x - x_N)$$

P_{N+1} = the Legendre polynomial of degree $N+1$

$\frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!}$ is required to normalize the leading coefficient of $P_{N+1}(x)$

- What have we done by defining $p_N(x)$ in this way $\Rightarrow$ we have selected the integration/interpolation/data points $x_o, x_1, \ldots x_N$ to be the ***roots*** of $P_{N+1}(x)$.

- In general

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}$$

$$P_o(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\vdots$$

- So far we have established
  - Selecting $p_N(x)$ to be proportional to the Legendre Polynomial of degree $N+1$ $\Rightarrow$ this satisfies the orthogonality condition which will lead to:

$$\int_{-1}^{+1} \beta_i(x)dx = 0$$

As a result $f_i^{(1)}$ terms will ***not*** appear in the Gauss-Legendre integration formula.

  - If we select $p_N(x)$ to be the Legendre Polynomial of degree $N+1$ $\Rightarrow$ the roots of that polynomial will represent the interpolating/integration/data points since $p_N(x) = (x-x_o)(x-x_1)\ldots(x-x_N)$ has been set equal to $CP_{N+1}(x)$

- Now we must find the weights of the integration formula. Note that A_i will represent the weights!

$$A_i \equiv \int_{-1}^{+1} \alpha_i(x)dx \qquad \Rightarrow$$

$$A_i = \int_{-1}^{+1} t_i(x)\ l_{iN}(x)\ l_{iN}(x) dx$$

where

$$t_i(x) = 1 - (x - x_i)\ 2\ l_{iN}^{(1)}(x_i)$$

$$l_{iN}(x) = \frac{p_N(x)}{(x - x_i) p_N^{(1)}(x_i)}$$

$$p_N(x) = (x - x_o)\ldots(x - x_N)$$

and where $x_o, \ldots, x_N$ are the ***roots*** of the Legendre polynomial of degree $N + 1$ or

$$p_N(x) = \frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!} P_{N+1}(x)$$

Two point Gauss-Legendre integration

Develop a 2 point Gauss-Legendre integration formula for $[-1, +1]$. Let

$$g(x) = \sum_{i=0}^{1} \alpha_i(x) f_i + \sum_{i=0}^{1} \beta_i(x) f_i^{(1)}$$

$$g(x) = \alpha_o(x) f_o + \alpha_1(x) f_1 + \beta_o(x) f_o^{(1)} + \beta_1(x) f_1^{(1)}$$

- Thus

$$I = \int_{-1}^{+1} g(x) dx + E \quad \Rightarrow$$

$$I = \int_{-1}^{+1} \alpha_o(x) f_o dx + \int_{-1}^{+1} \alpha_1(x) f_1 dx + \int_{-1}^{+1} \beta_o(x) f_o^{(1)} dx + \int_{-1}^{+1} \beta_1(x) f_1^{(1)} dx \quad \Rightarrow$$

$$I = f_o \int_{-1}^{+1} \alpha_o(x) dx + f_1 \int_{-1}^{+1} \alpha_1(x) dx + f_o^{(1)} \int_{-1}^{+1} \beta_o(x) dx + f_1^{(1)} \int_{-1}^{+1} \beta_1(x) dx$$

Step 1 - Establish interpolating points

- Interpolation points will be the roots of the Legendre Polynomial of order 2.

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow$$

$$\frac{1}{2}(3x^2 - 1) = 0 \Rightarrow$$

$$3x^2 = 1 \Rightarrow$$

$$x^2 = \frac{1}{3} \Rightarrow$$

$$x_{0,1} = \pm\sqrt{\frac{1}{3}} \Rightarrow$$

$$x_{0,1} = \pm 0.57735$$

- Checking these roots

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2(x^2-1)^2}{dx^2} \Rightarrow$$

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 + 1) \Rightarrow$$

$$P_2(x) = \frac{1}{8}(12x^2 - 4) \Rightarrow$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$p_1(x) = \frac{2^2(2!)^2}{(2(2))!} P_2(x) \Rightarrow$$

$$p_1(x) = \frac{4 \cdot 4}{4 \cdot 3 \cdot 2} \cdot \frac{1}{2}(3x^2 - 1) \Rightarrow$$

$$p_1(x) = x^2 - \frac{1}{3}$$

- From formula which defines $p_1(x)$ using the integration points

$$p_1(x) = \left(x + \sqrt{\frac{1}{3}}\right)\left(x - \sqrt{\frac{1}{3}}\right) = x^2 - \frac{1}{3}$$

Step 2 - Establish the coefficients of the derivative terms in the integration formula

- Let's demonstrate that with the roots $x_{o,1} = \pm 0.57735$ we will satisfy

$$\int_{-1}^{+1} \beta_o(x)dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x)dx = 0$$

- First develop $\beta_o(x)$ and $\beta_1(x)$ by developing $p_1(x)$, $p_1^{(1)}(x)$, $l_{o1}(x)$, $l_{11}(x)$, $s_o(x)$ and $s_1(x)$

$$p_1(x) = (x - x_o)(x - x_1)$$

$$p_1^{(1)}(x) = (x - x_o) + (x - x_1)$$

$$l_{j1}(x) = \frac{p_1(x)}{(x-x_j)p_1^{(1)}(x_j)} \qquad j = 0, 1 \quad \Rightarrow$$

$$l_{j1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_j)[(x_j-x_o)+(x_j-x_1)]}$$

$$l_{o1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_o)[x_o-x_o+x_o-x_1]} \quad \Rightarrow$$

$$l_{o1}(x) = \frac{x-x_1}{x_o-x_1}$$

$$l_{11}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_1)[(x_1-x_o)+(x_1-x_1)]} \quad \Rightarrow$$

$$l_{11}(x) = \frac{x-x_o}{x_1-x_o}$$

$$s_o(x) = x - x_o$$

$$s_1(x) = x - x_1$$

- Now we can establish $\beta_o(x)$

$$\beta_o(x) = s_o(x)\ l_{o1}(x)\ l_{o1}(x) \qquad \Rightarrow$$

$$\beta_o(x) = (x - x_o)\left(\frac{x - x_1}{x_o - x_1}\right)\left(\frac{x - x_1}{x_o - x_1}\right)$$

- Noting that $x_o = -\sqrt{\frac{1}{3}}$, $x_1 = \sqrt{\frac{1}{3}}$

$$\beta_o(x) = \left(x + \sqrt{\frac{1}{3}}\right)\frac{\left(x - \sqrt{\frac{1}{3}}\right)\left(x - \sqrt{\frac{1}{3}}\right)}{\left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}\right)^2} \qquad \Rightarrow$$

$$\beta_o(x) = \frac{3}{4}\left[x^3 - \sqrt{\frac{1}{3}}x^2 - \frac{1}{3}x + \left(\frac{1}{3}\right)^{3/2}\right]$$

- Similarly for $\beta_1(x)$

$$\beta_1(x) = s_1(x)\ l_{11}(x)\ l_{11}(x) \qquad \Rightarrow$$

$$\beta_1(x) = (x - x_1)\frac{(x - x_o)}{(x_1 - x_o)} \cdot \frac{(x - x_o)}{(x_1 - x_o)}$$

- Substituting $x_o = -\sqrt{\frac{1}{3}}$, $x_1 = \sqrt{\frac{1}{3}}$

$$\beta_1(x) = \frac{\left(x - \sqrt{\frac{1}{3}}\right)\left(x + \sqrt{\frac{1}{3}}\right)\left(x + \sqrt{\frac{1}{3}}\right)}{\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}\right)^2} \qquad \Rightarrow$$

$$\beta_1(x) = \frac{3}{4}\left[x^3 + \sqrt{\frac{1}{3}}x^2 - \frac{1}{3}x - \left(\frac{1}{3}\right)^{3/2}\right]$$

- Now we can develop $\int_{-1}^{+1} \beta_o(x)dx$

$$\int_{-1}^{+1} \beta_o(x)dx = \int_{-1}^{+1} \frac{3}{4}\left[x^3 - \sqrt{\frac{1}{3}}x^2 - \frac{1}{3}x + \left(\frac{1}{3}\right)^{3/2}\right]dx \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x)dx = \frac{3}{4}\left[\frac{x^4}{4} - \left(\frac{1}{3}\right)^{3/2}x^3 - \frac{1}{6}x^2 + \left(\frac{1}{3}\right)^{3/2}x\right]_{-1}^{+1} \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x)dx = \frac{3}{4}\left[\left(\frac{1}{4} - \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} + \left(\frac{1}{3}\right)^{3/2}\right) - \left(\frac{1}{4} + \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} - \left(\frac{1}{3}\right)^{3/2}\right)\right] \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x)dx = 0$$

• Develop $\int_{-1}^{+1} \beta_1(x)dx$

$$\int_{-1}^{+1} \beta_1(x)dx = \int_{-1}^{+1} \frac{3}{4}\left[x^3 + \sqrt{\frac{1}{3}}x^2 - \frac{1}{3}x - \left(\frac{1}{3}\right)^{3/2}\right]dx \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x)dx = \frac{3}{4}\left[\frac{x^4}{4} + \left(\frac{1}{3}\right)^{3/2}x^3 - \frac{1}{6}x^2 - \left(\frac{1}{3}\right)^{3/2}x\right]_{-1}^{+1} \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x)dx = \frac{3}{4}\left[\left(\frac{1}{4} + \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} - \left(\frac{1}{3}\right)^{3/2}\right) - \left(\frac{1}{4} - \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} + \left(\frac{1}{3}\right)^{3/2}\right)\right] \quad \Rightarrow$$

$$\int_{-1}^{+1} \beta_1(x)dx = 0$$

- Now our integration formula reduces to:

$$I = f_o \int_{-1}^{+1} \alpha_o(x)dx + f_1 \int_{-1}^{+1} \alpha_1(x)dx \quad \Rightarrow$$

$$I = A_o f_o + A_1 f_1$$

where

$$A_o \equiv \int_{-1}^{+1} \alpha_o(x)dx \quad \text{and} \quad A_1 \equiv \int_{-1}^{+1} \alpha_1(x)dx$$

Step 3 - Develop A_o, A_1

- Establish $\alpha_o(x)$

$$\alpha_o(x) = t_o(x)\ l_{o1}(x)\ l_{o1}(x) \quad \Rightarrow$$

$$\alpha_o(x) = [1 - (x - x_o)2l_{o1}^{(1)}(x_o)\ ]l_{o1}(x)l_{o1}(x) \quad \Rightarrow$$

$$\alpha_o(x) = \left\{1 - (x - x_o)\left[\frac{2}{x_o - x_1}\right]\right\}\left(\frac{x - x_1}{x_o - x_1}\right)\left(\frac{x - x_1}{x_o - x_1}\right) \Rightarrow$$

$$\alpha_o(x) = \left\{1 - \left(x + \sqrt{\frac{1}{3}}\right)\left[\frac{2}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}}\right]\right\}\frac{\left(x - \sqrt{\frac{1}{3}}\right)\left(x - \sqrt{\frac{1}{3}}\right)}{\left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}\right)\left(-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}\right)} \Rightarrow$$

$$\alpha_o(x) = \frac{3}{4}\left\{1 - \left(x + \sqrt{\frac{1}{3}}\right)\left(\frac{2}{-2\sqrt{\frac{1}{3}}}\right)\right\}\left(x^2 - 2\sqrt{\frac{1}{3}}x + \frac{1}{3}\right) \Rightarrow$$

$$\alpha_o(x) = \frac{3\sqrt{3}}{4}\left\{\frac{2}{\sqrt{3}} + x\right\}\left(x^2 - 2\sqrt{\frac{1}{3}}x + \frac{1}{3}\right) \Rightarrow$$

$$\alpha_o(x) = \frac{3}{4}\sqrt{3}\left\{x^3 - x + 2\left(\frac{1}{3}\right)^{3/2}\right\}$$

- Develop $\int_{-1}^{+1} \alpha_o(x)dx$

$$\int_{-1}^{+1} \alpha_o(x)dx = \int_{-1}^{+1} \left[\frac{3}{4}\sqrt{3}\left(x^3 - x + 2\left(\frac{1}{3}\right)^{3/2}\right)\right]dx \quad \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x)dx = \frac{3}{4}\sqrt{3}\left[\frac{x^4}{4} - \frac{x^2}{2} + 2\left(\frac{1}{3}\right)^{3/2} x\right]_{-1}^{+1} \quad \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x)dx = \frac{3}{4}\sqrt{3}\left[\left(\frac{1}{4} - \frac{1}{2} + 2\left(\frac{1}{3}\right)^{3/2}\right) - \left(\frac{1}{4} - \frac{1}{2} - 2\left(\frac{1}{3}\right)^{3/2}\right)\right] \quad \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x)dx = \frac{3}{4}\sqrt{3}\left(4\sqrt{\frac{1}{3}}\ \frac{1}{3}\right) \quad \Rightarrow$$

$$\int_{-1}^{+1} \alpha_o(x)dx = 1 \quad \Rightarrow$$

$$\boldsymbol{A_o = 1}$$

- Similarly we can show that $A_1 = \int_{-1}^{+1} \alpha_1(x)dx = 1$

- **Thus we have established the two point Gauss Quadrature rule**

$$I = \int_{-1}^{+1} f(x)dx = w_o f_o + w_1 f_1$$

where $x_o = -\sqrt{\frac{1}{3}}$ **and** $x_1 = +\sqrt{\frac{1}{3}}$ **are the integration points and** $w_o = w_1 = 1$

- We note that this integration rule was established by defining a Hermite cubic interpolating function and defining the integration points x_o, x_1 such that

$$\int_{-1}^{+1} \beta_o(x)dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x)dx = 0$$

  - Therefore the functional derivative values drop out of the Gauss Legendre integration formula!