LEcTURe 16

GAUSS QUADRATURE

• In general for Newton-Cotes (equispaced interpolation points/ data points/ integration points/ nodes).

\[ \int_{x_S}^{x_E} f(x) dx = h \left[ w_0 f_0 + w_1 f_1 + \ldots + w_N f_N \right] + E \]

• Note that for Newton-Cotes formulae only the weighting coefficients \( w_i \) were unknown and the \( x_i \) were fixed.
• However the number of and placement of the integration points influences the accuracy of the Newton-Cotes formulae:

  • $N$ even $\rightarrow N^{th}$ degree interpolation function exactly integrates an $N + 1^{th}$ degree polynomial $\rightarrow$ This is due to the placement of one of the data points.

  • $N$ odd $\rightarrow N^{th}$ degree interpolation function exactly integrates an $N^{th}$ degree polynomial.

• Concept: Let’s allow the placement of the integration points to vary such that we further increase the degree of the polynomial we can integrate exactly for a given number of integration points.

• In fact we can integrate an $2N + 1$ degree polynomial exactly with only $N + 1$ integration points
• Assume that for Gauss Quadrature the form of the integration rule is

\[ \int_{x_S}^{x_E} f(x) \, dx = [w_0 f_o + w_1 f_1 + \ldots + w_N f_N] + E \]

• In *deriving* (not applying) these integration formulae
  • Location of the integration points, \( x_i \), \( i = O, N \) are unknown
  • Integration formulae weights, \( w_i \), \( i = O, N \) are unknown

• 2\( (N + 1) \) unknowns → we will be able to exactly integrate any 2\( N + 1 \) degree polynomial!
Derivation of Gauss Quadrature by Integrating Exact Polynomials and Matching

Derive 1 point Gauss-Quadrature

• 2 unknowns \( w_o, x_o \) which will exactly integrate any linear function

• Let the general polynomial be

\[
f(x) = Ax + B
\]

where the coefficients \( A, B \) can equal any value

• Also consider the integration interval to be \([-1, +1]\) such that \( x_S = -1 \) and \( x_E = +1 \) (no loss in generality since we can always transform coordinates).

\[
\int_{-1}^{+1} f(x) \, dx = w_o \, f(x_o)
\]

• Substituting in the form of \( f(x) \)

\[
\int_{-1}^{+1} (Ax + B) \, dx = w_o (Ax_o + B) \quad \Rightarrow
\]
• In order for this to be true for any 1st degree polynomial (i.e. any \( A \) and \( B \)).

\[
\left[ \frac{Ax^2}{2} + Bx \right]_{-1}^{+1} = w_o(Ax_o + B) \Rightarrow \\
A(0) + B(2) = A(x_o w_o) + B(w_o)
\]

• Therefore \( x_o = 0 \), \( w_o = 2 \) for 1 point \((N = 1)\) Gauss Quadrature.

• We can integrate exactly with only 1 point for a linear function while for Newton-Cotes we needed two points!
Derive a 2 point Gauss Quadrature Formula

- The general form of the integration formula is
  \[ I = w_o f_o + w_1 f_1 \]
  - \( w_o, x_o, w_1, x_1 \) are all unknowns
  - 4 unknowns \( \Rightarrow \) we can fit a 3rd degree polynomial exactly
    \[ f(x) = Ax^3 + Bx^2 + Cx + D \]
  - Substituting in for \( f(x) \) into the general form of the integration rule
    \[
    \int_{-1}^{+1} f(x) \, dx = w_o f(x_o) + w_1 f(x_1)
    \]
    \( \Rightarrow \)
• In order for this to be true for any third degree polynomial (i.e. all arbitrary coefficients, $A$, $B$, $C$, $D$), we must have:

\[
\begin{align*}
w_o x_o^3 + w_1 x_1^3 &= 0 \\
w_o x_o^2 + w_1 x_1^2 - \frac{2}{3} &= 0 \\
w_o x_o + w_1 x_1 &= 0 \\
w_o + w_1 - 2 &= 0
\end{align*}
\]
• 4 nonlinear equations $\rightarrow$ 4 unknowns

$$w_o = 1 \quad \text{and} \quad w_1 = 1$$

$$x_o = -\frac{1}{\sqrt{3}} \quad \text{and} \quad x_1 = +\frac{1}{\sqrt{3}}$$

• All polynomials of degree 3 or less will be exactly integrated with a Gauss-Legendre 2 point formula.
**Gauss Legendre Formulae**

\[
    I = \int_{-1}^{+1} f(x) \, dx = \sum_{i=0}^{N} w_i f_i + E
\]

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<th>(N)</th>
<th>(N + 1)</th>
<th>(x_i, \quad i = 0, N)</th>
<th>(w_i)</th>
<th>Exact for polynomials of degree</th>
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• Notes

• \( N + 1 \) = the number of integration points

• Integration points are symmetrical on \([-1, +1]\)

• Formulae can be applied on any interval using a coordinate transformation

• \( N + 1 \) integration points \( \rightarrow \) will integrate polynomials of up to degree \( 2N + 1 \) exactly.

  • Recall that Newton Cotes \( \rightarrow \) \( N + 1 \) integration points only integrates an \( N^{th}/N + 1^{th} \) degree polynomial exactly depending on \( N \) being odd or even.

  • For Gauss-Legendre integration, we allowed both weights and integration point locations to vary to match an integral exactly \( \Rightarrow \) more d.o.f. \( \Rightarrow \) allows you to match a higher degree polynomial!

  • An alternative way of looking at Gauss-Legendre integration formulae is that we use Hermite interpolation instead of Lagrange interpolation! (How can this be since Hermite interpolation involves derivatives \( \Rightarrow \) let’s examine this!)
Derivation of Gauss Quadrature by Integrating Hermite Interpolating Functions

**Hermite interpolation formulae**

- Hermite interpolation which *matches* the function and the first derivative at \( N + 1 \) interpolation points is expressed as:

\[
  g(x) = \sum_{i=0}^{N} \alpha_i(x) f_i + \sum_{i=0}^{N} \beta_i(x) f_i^{(1)}
\]

- It can be shown that in general for non-equispaced points

\[
  \alpha_i(x) = t_i(x) l_{iN}(x) l_{iN}(x) \quad i = 0, N
\]

\[
  \beta_i(x) = s_i(x) l_{iN}(x) l_{iN}(x) \quad i = 0, N
\]
where

\( p_N(x) \equiv (x - x_0)(x - x_1) \cdots (x - x_N) \)

\[ l_{iN}(x) \equiv \frac{p_N(x)}{(x - x_i)_{p_N}^{(1)}(x_i)} \quad i = 0, N \]

\[ t_i(x) \equiv 1 - (x - x_i) \cdot 2 \cdot l_{iN}^{(1)}(x_i) \]

\[ s_i(x) \equiv (x - x_i) \]
Example of defining a cubic Hermite interpolating function

• Derive Hermite interpolating functions for 2 interpolation points located at \(-1\) and \(+1\) for the interval \([-1, +1]\).

\[ f_0 f_0^{(1)} \] \hspace{1cm} \[ f_1 f_1^{(1)} \]
\[ x_0 = -1 \] \hspace{1cm} \[ x_1 = +1 \]

\[ N + 1 = 2 \] \hspace{0.5cm} \text{points} \quad \Rightarrow \quad N = 1

• Establish \( p_N(x) \)

\[ p_1(x) = (x - x_o)(x - x_1) \quad \Rightarrow \]

\[ p_1^{(1)}(x) = (x - x_o) + (x - x_1) \]
• Establish $l_{iN}(x)$

$$l_{i1}(x) = \frac{p_1(x)}{(x-x_i)[p_1'(x_i)]}$$

$$l_{i1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_i)[(x_i-x_o)+(x_i-x_1)]}$$

• Let $i = 0$

$$l_{o1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_o)[0+(x_o-x_1)]} \Rightarrow$$

$$l_{o1}(x) = \frac{x-x_1}{x_o-x_1}$$

• Substitute in $x_o = -1$ and $x_1 = +1$

$$l_{o1}(x) = \frac{1}{2}(1-x)$$
• Let $i = 1$

$$l_{11}(x) = \frac{(x - x_o)(x - x_1)}{(x - x_1)[(x_1 - x_o) + 0]}$$

• Substitute in values for $x_o, x_1$

$$l_{11}(x) = \frac{1}{2}(1 + x)$$

• Taking derivatives

$$l_{o1}^{(1)}(x) = -\frac{1}{2}$$

$$l_{11}^{(1)}(x) = +\frac{1}{2}$$
• Establish \( t_i(x) \)

\[
t_o(x) = 1 - (x - x_o)2l^{(1)}_{o1}(x_o) \quad \Rightarrow
\]

\[
t_o(x) = 1 - (x + 1)(2\left(-\frac{1}{2}\right)) \quad \Rightarrow
\]

\[
t_o(x) = 2 + x
\]

\[
t_1(x) = 1 - (x - x_1)2l^{(1)}_{11}(x_1) \quad \Rightarrow
\]

\[
t_1(x) = 1 - (x - 1)\left(2 \cdot \frac{1}{2}\right) \quad \Rightarrow
\]

\[
t_1(x) = 2 - x
\]

• Establish \( s_i(x) \)

\[
s_o(x) = x + 1
\]

\[
s_1(x) = x - 1
\]
• Establish $\alpha_i(x)$

\[
\alpha_o(x) = t_o(x) l_{o1}(x) l_{o1}(x) \quad \Rightarrow
\]

\[
\alpha_o(x) = (2 + x) \frac{1}{2} (1 - x) \frac{1}{2} (1 - x) \quad \Rightarrow
\]

\[
\alpha_o(x) = \frac{1}{4} (2 - 3x + x^3)
\]

\[
\alpha_1(x) = t_1(x) l_{11}(x) l_{11}(x) \quad \Rightarrow
\]

\[
\alpha_1(x) = (2 - x) \frac{1}{2} (1 + x) \frac{1}{2} (1 + x) \quad \Rightarrow
\]

\[
\alpha_1(x) = \frac{1}{4} (2 + 3x - x^3)
\]
• Establish $\beta_i(x)$

\[
\beta_0(x) = s_0(x)l_{o1}(x)l_{o1}(x) \quad \Rightarrow \\
\beta_0(x) = (x + 1)\frac{1}{2}(1 - x)\frac{1}{2}(1 - x) \quad \Rightarrow \\
\beta_0(x) = \frac{1}{4}(1 - x - x^2 + x^3) \\
\beta_1(x) = s_1(x)l_{11}(x)l_{11}(x) \quad \Rightarrow \\
\beta_1(x) = (x - 1)\frac{1}{2}(1 + x)\frac{1}{2}(1 + x) \quad \Rightarrow \\
\beta_1(x) = \frac{1}{4}(-1 - x + x^2 + x^3)
\]

• In general

\[
g(x) = \alpha_0(x)f_0 + \alpha_1(x)f_1 + \beta_0(x)f_0^{(1)} + \beta_1(x)f_1^{(1)}
\]
• These functions satisfy the constraints

\[ \alpha_i(x_j) = \delta_{ij} \]
\[ \beta_i(x_j) = 0 \]
Gauss-Legendre Quadrature by integrating Hermite interpolating polynomials

\[ I = \int_{-1}^{+1} f(x)dx = \sum_{i=0}^{N} w_i f_i + E \]

- **Notes**
  - Use \([-1, +1]\) without loss of generality \(\Rightarrow\) we can always transform the interval.
  - Approximation for \(I\) is exact for \(2N+1\) degree polynomials
- We can derive all Gauss-Legendre quadrature formulae by approximating \(f(x)\) with an \(2N+1^{th}\) degree Hermite interpolating function using \(N\) specially selected integration/interpolation points.

\[ I = \int_{-1}^{+1} g(x)dx + E \]

where

\[ g(x) = \sum_{i=0}^{N} \alpha_i(x)f_i + \sum_{i=0}^{N} \beta_i(x)f_i^{(1)} \]
• Thus

\[
I = \int_{-1}^{1} \left[ \sum_{i=0}^{N} \alpha_i(x)f_i + \sum_{i=0}^{N} \beta_i(x)f_i^{(1)}^i \right] dx + E
\]

\[\Rightarrow\]

\[
I = \sum_{i=0}^{N} A_i f_i + \sum_{i=0}^{N} B_i f_i^{(1)} + E
\]

where

\[
A_i \equiv \int_{-1}^{1} \alpha_i(x)dx \quad \text{and} \quad B_i \equiv \int_{-1}^{1} \beta_i(x)dx
\]

• Furthermore we can show that

\[
E = \int_{-1}^{1} \left[ \frac{P_{N+1}^2(x)}{(2N+2)!} f^{(2N+2)}(x) + \text{H.O.T.} \right] dx
\]
• Note that we are assuming Taylor series expansions about \( x_0 \) and using higher order terms in the expansion.

  • Therefore \( E = 0 \) for any polynomial of degree \( 2N + 1 \) or less!

• The problem that we encounter is that the integration formula as it now stands in general requires us to know both functional and first derivative values at the nodes!

• Let us select \( x_0, x_1, x_2, \ldots x_N \) such that

\[
B_i = 0 \quad i = 0, N \quad \Rightarrow \\
+1 \\
\int \beta_i(x) \, dx = 0 \quad i = 0, N \quad \Rightarrow \\
-1 \\
+1 \\
\int s_i(x) l_{iN}(x) l_{iN}(x) \, dx = 0 \quad i = 0, N \quad \Rightarrow \\
-1 \\
+1 \\
\int (x - x_i) \frac{p_N(x)}{(x - x_i) p_N^{(1)}(x_i)} l_{iN}(x) \, dx = 0 \quad i = 0, N \quad \Rightarrow \\
-1
\]
Therefore we require $p_N(x)$ to be orthogonal on $[-1, +1]$ to all polynomials of degree $N$ or less ⇒ any multiple of Legendre-Polynomials will satisfy this.

Let

$$p_N(x) = \frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!} P_{N+1}(x)$$

where

$$p_N(x) = (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_N)$$

$P_{N+1}$ = the Legendre polynomial of degree $N+1$

$$\frac{2^{N+1}[(N+1)!]^2}{[2(N+1)]!}$$

is required to normalize the leading coefficient of $P_{N+1}(x)$
• What have we done by defining $p_N(x)$ in this way $\Rightarrow$ we have selected the integration/interpolation/data points $x_0, x_1, \ldots x_N$ to be the roots of $P_{N+1}(x)$.

• In general

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n(x^2 - 1)^n}{dx^n}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

\[ \cdots \]

\[ \cdots \]
• So far we have established

• Selecting \( p_N(x) \) to be proportional to the Legendre Polynomial of degree \( N + 1 \) \( \Rightarrow \) this satisfies the orthogonality condition which will lead to:

\[
\int_{-1}^{1} \beta_i(x) dx = 0
\]

As a result \( f_i^{(1)} \) terms will not appear in the Gauss-Legendre integration formula.

• If we select \( p_N(x) \) to be the Legendre Polynomial of degree \( N + 1 \) \( \Rightarrow \) the roots of that polynomial will represent the interpolating/integration/data points since \( p_N(x) = (x - x_0)(x - x_1)\ldots(x - x_N) \) has been set equal to \( CP_{N+1}(x) \)

• Now we must find the weights of the integration formula. Note that \( A_i \) will represent the weights!

\[
A_i = \int_{-1}^{1} \alpha_i(x) dx \quad \Rightarrow
\]

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where

\[ A_i = \int_{-1}^{+1} t_i(x) l_{iN}(x) l_{iN}(x) dx \]

where

\[ t_i(x) = 1 - (x - x_i) 2 l_{iN}^{(1)}(x_i) \]

\[ l_{iN}(x) = \frac{p_{N}(x)}{(x - x_i) p_{N}^{(1)}(x_i)} \]

\[ p_N(x) = (x - x_o) \ldots (x - x_N) \]

and where \( x_o, \ldots, x_N \) are the \textbf{roots} of the Legendre polynomial of degree \( N + 1 \) or

\[ p_N(x) = \frac{2^{N+1}[(N + 1)!]^2}{[2(N + 1)]!} P_{N+1}(x) \]
**Two point Gauss-Legendre integration**

Develop a 2 point Gauss-Legendre integration formula for \([-1, +1]\). Let

\[
g(x) = \sum_{i=0}^{1} \alpha_i(x)f_i + \sum_{i=0}^{1} \beta_i(x)f_i^{(1)}
\]

\[
g(x) = \alpha_o(x)f_o + \alpha_1(x)f_1 + \beta_o(x)f_o^{(1)} + \beta_1(x)f_1^{(1)}
\]

- Thus

\[
I = \int_{-1}^{+1} g(x)dx + E \quad \Rightarrow
\]

\[
I = \int_{-1}^{+1} \alpha_o(x)f_o dx + \int_{-1}^{+1} \alpha_1(x)f_1 dx + \int_{-1}^{+1} \beta_o(x)f_o^{(1)} dx + \int_{-1}^{+1} \beta_1(x)f_1^{(1)} dx \quad \Rightarrow
\]

\[
I = f_o \int_{-1}^{+1} \alpha_o(x)dx + f_1 \int_{-1}^{+1} \alpha_1(x)dx + f_o^{(1)} \int_{-1}^{+1} \beta_o(x)dx + f_1^{(1)} \int_{-1}^{+1} \beta_1(x)dx
\]
Step 1 - Establish interpolating points

- Interpolation points will be the roots of the Legendre Polynomial of order 2.

\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow \]

\[ \frac{1}{2}(3x^2 - 1) = 0 \Rightarrow \]

\[ 3x^2 = 1 \Rightarrow \]

\[ x^2 = \frac{1}{3} \Rightarrow \]

\[ x_{0,1} = \pm \sqrt{\frac{1}{3}} \Rightarrow \]

\[ x_{0,1} = \pm 0.57735 \]
• Checking these roots

\[ P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2(x^2 - 1)^2}{dx^2} \Rightarrow \]

\[ P_2(x) = \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) \Rightarrow \]

\[ P_2(x) = \frac{1}{8} (12x^2 - 4) \Rightarrow \]

\[ P_2(x) = \frac{1}{2} (3x^2 - 1) \]

\[ p_1(x) = \frac{2^2 (2!)^2}{(2(2))!} P_2(x) \Rightarrow \]

\[ p_1(x) = \frac{4 \cdot 4}{4 \cdot 3 \cdot 2} \cdot \frac{1}{2} (3x^2 - 1) \Rightarrow \]

\[ p_1(x) = x^2 - \frac{1}{3} \]
• From formula which defines $p_1(x)$ using the integration points

$$p_1(x) = \left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) = x^2 - \frac{1}{3}$$

**Step 2 - Establish the coefficients of the derivative terms in the integration formula**

• Let’s demonstrate that with the roots $x_{o,1} = \pm 0.57735$ we will satisfy

$$\int_{-1}^{+1} \beta_o(x)dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x)dx = 0$$

• First develop $\beta_o(x)$ and $\beta_1(x)$ by developing $p_1(x), p_1^{(1)}(x), l_{o1}(x), l_{11}(x), s_o(x)$ and $s_1(x)$

$$p_1(x) = (x-x_o)(x-x_1)$$

$$p_1^{(1)}(x) = (x-x_o) + (x-x_1)$$
\[ l_{j1}(x) = \frac{p_1(x)}{(x-x_j)p_1^{(1)}(x_j)} \quad j = 0, 1 \Rightarrow \]

\[ l_{j1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_j)[(x_j-x_o) + (x_j-x_1)]} \]

\[ l_{o1}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_o)[x_o-x_o + x_o-x_1]} \Rightarrow \]

\[ l_{o1}(x) = \frac{x-x_1}{x_o-x_1} \]

\[ l_{11}(x) = \frac{(x-x_o)(x-x_1)}{(x-x_1)[(x_1-x_o) + (x_1-x_1)]} \Rightarrow \]

\[ l_{11}(x) = \frac{x-x_o}{x_1-x_o} \]
Now we can establish $\beta_o(x)$

$$\beta_o(x) = s_o(x) l_{o1}(x) l_{o1}(x)$$

$$\beta_o(x) = (x - x_o) \left( \frac{x - x_1}{x_o - x_1} \right) \left( \frac{x - x_1}{x_o - x_1} \right)$$

Noting that $x_o = -\frac{1}{\sqrt{3}}$, $x_1 = \frac{1}{\sqrt{3}}$

$$\beta_o(x) = \left( x + \frac{1}{\sqrt{3}} \right) \frac{\left( x - \frac{1}{\sqrt{3}} \right) \left( x - \frac{1}{\sqrt{3}} \right)}{-\left( \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right)^2}$$

$$\beta_o(x) = \frac{3}{4} \left[ x^3 - \frac{1}{\sqrt{3}} x^2 - \frac{1}{3} x + \left( \frac{1}{3} \right)^{3/2} \right]$$
• Similarly for $\beta_1(x)$

$$\beta_1(x) = s_1(x) \cdot l_{11}(x) \cdot l_{11}(x) \Rightarrow$$

$$\beta_1(x) = (x - x_1) \frac{(x - x_o)}{(x_1 - x_o)} \cdot \frac{(x - x_o)}{(x_1 - x_o)}$$

• Substituting $x_o = -\frac{1}{\sqrt[3]{3}}, x_1 = \frac{1}{\sqrt[3]{3}}$

$$\beta_1(x) = \frac{(x - \frac{1}{\sqrt[3]{3}})(x + \frac{1}{\sqrt[3]{3}})(x + \frac{1}{\sqrt[3]{3}})}{\left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{3}}\right)^2} \Rightarrow$$

$$\beta_1(x) = \frac{3}{4}\left[x^3 + \sqrt[3]{\frac{1}{3}}x^2 - \frac{1}{3}x - \left(\frac{1}{3}\right)^{3/2}\right]$$
Now we can develop $\int_{-1}^{+1} \beta_o(x) dx$

$$\int_{-1}^{+1} \beta_o(x) dx = \int_{-1}^{+1} \frac{3}{4} \left[ x^3 - \sqrt[3]{\frac{1}{3}} x^2 - \frac{1}{3} x + \left(\frac{1}{3}\right)^{3/2} \right] dx \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = \frac{3}{4} \left[ \left(\frac{1}{4}\right)^{3/2} - \left(\frac{1}{3}\right)^{3/2} \right] - \frac{1}{6} \left(\frac{1}{3}\right)^{3/2} x \right]^{+1}_{-1} \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = \frac{3}{4} \left[ \left(\frac{1}{4}\right)^{3/2} - \left(\frac{1}{3}\right)^{3/2} \right] - \frac{1}{6} \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} \left(\frac{1}{3}\right)^{3/2} \right] \Rightarrow$$

$$\int_{-1}^{+1} \beta_o(x) dx = 0$$
• Develop $\int_{-1}^{+1} \beta_1(x) \, dx$

\[
\int_{-1}^{+1} \beta_1(x) \, dx = \int_{-1}^{+1} \frac{3}{4} \left[ x^3 + \sqrt[3]{\frac{1}{3}}x^2 - \frac{1}{3}x - \left(\frac{1}{3}\right)^{3/2} \right] \, dx \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \beta_1(x) \, dx = \frac{3}{4} \left[ \frac{x^4}{4} + \left(\frac{1}{3}\right)^{3/2} x^3 - \frac{1}{6}x^2 - \left(\frac{1}{3}\right)^{3/2} x \right]_{-1}^{+1} \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \beta_1(x) \, dx = \frac{3}{4} \left[ \left(\frac{1}{4} + \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} - \left(\frac{1}{3}\right)^{3/2} \right) - \left(\frac{1}{4} - \left(\frac{1}{3}\right)^{3/2} - \frac{1}{6} + \left(\frac{1}{3}\right)^{3/2} \right) \right] \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \beta_1(x) \, dx = 0
\]
• Now our integration formula reduces to:

\[
I = f_o \int_{-1}^{+1} \alpha_o(x) \, dx + f_1 \int_{-1}^{+1} \alpha_1(x) \, dx \quad \Rightarrow \\
I = A_o f_o + A_1 f_1
\]

where

\[
A_o \equiv \int_{-1}^{+1} \alpha_o(x) \, dx \quad \text{and} \quad A_1 \equiv \int_{-1}^{+1} \alpha_1(x) \, dx
\]

**Step 3 - Develop** $A_o$, $A_1$

• Establish $\alpha_o(x)$

\[
\alpha_o(x) = t_o(x) \, l_{o1}(x) \, l_{o1}(x) \quad \Rightarrow \\
\alpha_o(x) = [1 - (x - x_o)2l_o^{(1)}(x_o)]l_{o1}(x)l_{o1}(x) \quad \Rightarrow
\]
\[ \alpha_o(x) = \left\{ 1 - (x - x_o) \left[ \frac{2}{x_o - x_1} \right] \left( \frac{x - x_1}{x_o - x_1} \right) \left( \frac{x - x_1}{x_o - x_1} \right) \right\} \Rightarrow \]

\[ \alpha_o(x) = \left\{ 1 - \left( x + \frac{1}{\sqrt[3]{3}} \right) \left[ \frac{2}{\sqrt[3]{-\frac{1}{3} - \frac{1}{3}}} \right] \left( \frac{x - \frac{1}{\sqrt[3]{3}}}{\sqrt[3]{-\frac{1}{3} - \frac{1}{3}}} \right) \left( \frac{x - \frac{1}{\sqrt[3]{3}}}{\sqrt[3]{-\frac{1}{3} - \frac{1}{3}}} \right) \right\} \Rightarrow \]

\[ \alpha_o(x) = \frac{3}{4} \left\{ 1 - \left( x + \frac{1}{\sqrt[3]{3}} \right) \left[ \frac{2}{\sqrt[3]{-2\frac{1}{\sqrt[3]{3}}}} \right] \right\} \left( x^2 - 2 \sqrt[3]{\frac{1}{3}x + \frac{1}{3}} \right) \Rightarrow \]

\[ \alpha_o(x) = \frac{3\sqrt[3]{3}}{4} \left\{ \frac{2}{\sqrt[3]{3}} + x \right\} \left( x^2 - 2 \sqrt[3]{\frac{1}{3}x + \frac{1}{3}} \right) \Rightarrow \]

\[ \alpha_o(x) = \frac{3}{4} \sqrt[3]{3} \left\{ x^3 - x + 2 \left( \frac{1}{3} \right)^{3/2} \right\} \]

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- Develop \( \int_{-1}^{+1} \alpha_o(x) \, dx \)

\[
\int_{-1}^{+1} \alpha_o(x) \, dx = \int_{-1}^{+1} \left[ \frac{3}{4} \sqrt{3} \left( x^3 - x + 2 \left( \frac{1}{3} \right)^{3/2} \right) \right] \, dx \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \alpha_o(x) \, dx = \frac{3}{4} \sqrt{3} \left[ \frac{x^4}{4} - \frac{x^2}{2} + \frac{2}{3} \right]_{-1}^{+1} \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \alpha_o(x) \, dx = \frac{3}{4} \sqrt{3} \left[ \left( \frac{1}{4} - \frac{1}{2} + \frac{2}{3} \right) - \left( \frac{1}{4} - \frac{1}{2} - \frac{2}{3} \right) \right] \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \alpha_o(x) \, dx = \frac{3}{4} \sqrt{3} \left( 4 \sqrt[3]{\frac{1}{3}} \right) \quad \Rightarrow
\]

\[
\int_{-1}^{+1} \alpha_o(x) \, dx = 1 \quad \Rightarrow
\]

\[
A_o = 1
\]
• Similarly we can show that \[ A_1 = \int_{-1}^{+1} \alpha_1(x) \, dx = 1 \]

• Thus we have established the two point Gauss Quadrature rule

\[ I = \int_{-1}^{+1} f(x) \, dx = w_o f_o + w_1 f_1 \]

where \( x_o = -\sqrt{\frac{1}{3}} \) and \( x_1 = +\sqrt{\frac{1}{3}} \) are the integration points and \( w_o = w_1 = 1 \)

• We note that this integration rule was established by defining a Hermite cubic interpolating function and defining the integration points \( x_o, x_1 \) such that

\[ \int_{-1}^{+1} \beta_o(x) \, dx = 0 \quad \text{and} \quad \int_{-1}^{+1} \beta_1(x) \, dx = 0 \]

• Therefore the functional derivative values drop out of the Gauss Legendre integration formula!