LECTURE 9

PRACTICAL ISSUES IN APPLYING FINITE DIFFERENCE APPROXIMATIONS

Derivatives of Variable Coefficients

- Consider the term

$$\frac{d}{dx}\left(g(x)\frac{df(x)}{dx}\right)$$

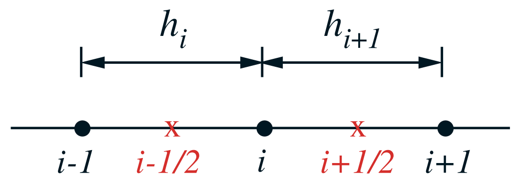
where g is a coefficient which is a function of x

- Proceed in steps. First let:

$$u \equiv g(x)\frac{df}{dx}$$

- Now evaluate:

$$\frac{du}{dx} = u_i^{(1)}$$



$$u_i^{(1)} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\frac{1}{2}(h_i + h_{i+1})}$$

- Now evaluate $u_{i+\frac{1}{2}}$ and $u_{i-\frac{1}{2}}$

$$u_{i+\frac{1}{2}} = g_{i+\frac{1}{2}} f^{(1)}_{i+\frac{1}{2}} = g_{i+\frac{1}{2}} \left\{ \frac{f_{i+1} - f_i}{h_{i+1}} \right\}$$

$$u_{i-\frac{1}{2}} = g_{i-\frac{1}{2}} f^{(1)}_{i-\frac{1}{2}} = g_{i-\frac{1}{2}} \left\{ \frac{f_i - f_{i-1}}{h_i} \right\}$$

- Hence

$$u_i^{(1)} = f_{i+1}\left[\frac{g_{i+1/2}}{h_{avg}h_{i+1}}\right] - f_i\left[\frac{g_{i+1/2}}{h_{avg}h_{i+1}} + \frac{g_{i-1/2}}{h_{avg}h_i}\right] + f_{i-1}\left[\frac{g_{i-1/2}}{h_{avg}h_i}\right]$$

where $h_{avg} \equiv \dfrac{h_i + h_{i+1}}{2}$

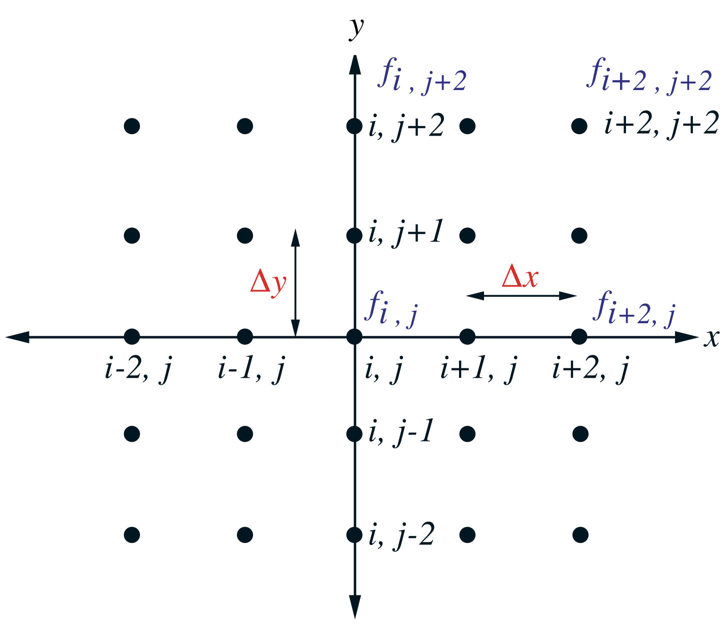
- When $h = h_{i+1} = h_i$, then the approximation reduces to:

$$u_i^{(1)} = \frac{1}{h^2}\left[f_{i+1}g_{i+\frac{1}{2}} - f_i\left(g_{i+\frac{1}{2}} + g_{i-\frac{1}{2}}\right) + f_{i-1}g_{i-\frac{1}{2}}\right]$$

- Example Applications with variable coefficients:
  - Fluids: spatially varying viscosities/diffusivities
  - Solids: spatially varying elasticities

Two Dimensional Finite Difference Approximations

- Define nodes in a two-dimensional plane with
  - i = spatial index in the x-direction
  - j = spatial index in the y-direction



- When taking partial derivatives w.r.t. x, we hold y constant

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

and

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta x^2}$$

- Similarly

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta y^2}$$

Example 1

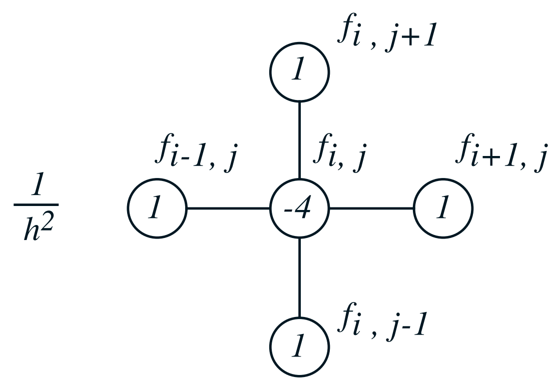
- Consider Laplace's equation and $\Delta x = \Delta y = h$:

$$\nabla^2 f\big|_{i,j} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow$$

$$\nabla^2 f\big|_{i,j} = \frac{1}{h^2}(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j})$$

- This may be viewed as follows in terms of a molecule:



Example 2

- Let's consider:

$$\nabla^4 f = \nabla^2(\nabla^2 f) = \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}$$

- Draw a molecule diagram for the 2nd order central difference formulation.

- Obtain $\frac{\partial^4 f}{\partial x^4}$ by finding $\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}\right)$

$$\frac{\partial^4 f}{\partial x^4} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^4} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{\Delta x^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j})\right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^2}\left(\frac{\partial^2}{\partial x^2}(f_{i+1,j}) - 2\frac{\partial^2}{\partial x^2}(f_{i,j}) + \frac{\partial^2}{\partial x^2}(f_{i-1,j})\right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^2}\left(\frac{1}{\Delta x^2}(f_{i+2,j} - 2f_{i+1,j} + f_{i,j})\right.$$

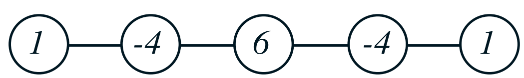
$$- \frac{2}{\Delta x^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j})$$

$$\left. + \frac{1}{\Delta x^2}(f_{i,j} - 2f_{i-1,j} + f_{i-2,j})\right)$$

$$\Rightarrow$$

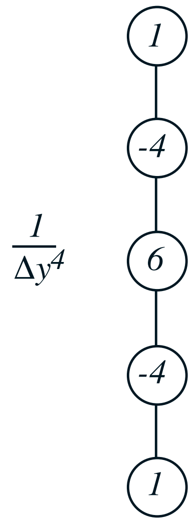
$$\frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^4}(f_{i-2,j} - 4f_{i-1,j} + 6f_{i,j} - 4f_{i+1,j} + f_{i+2,j})$$

$\frac{1}{\Delta x^4}$



- Similarly:

$$\frac{\partial^4 f}{dy^4} = \frac{1}{\Delta y^4}(f_{i,j-2} - 4f_{i,j-1} + 6f_{i,j} - 4f_{i,j+1} + f_{i,j+2})$$



- Evaluation of the mixed derivative:

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{1}{\Delta x^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j})\right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{1}{\Delta x^2}\left[\frac{1}{\Delta y^2}(f_{i-1,j-1} - 2f_{i-1,j} + f_{i-1,j+1})\right.$$

$$- \frac{2}{\Delta y^2}(f_{i,j-1} - 2f_{i,j} + f_{i,j+1})$$

$$\left. + \frac{1}{\Delta y^2}(f_{i+1,j-1} - 2f_{i+1,j} + f_{i+1,j+1})\right]$$

$$\Rightarrow$$

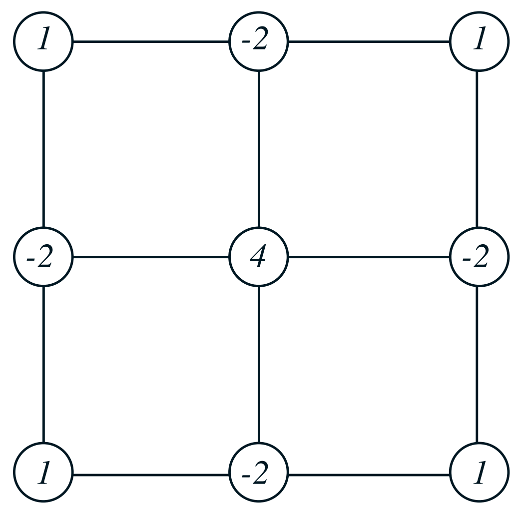
$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{1}{\Delta x^2 \Delta y^2}[f_{i-1,j-1} - 2f_{i-1,j} + f_{i-1,j+1}$$

$$-2f_{i,j-1} + 4f_{i,j} - 2f_{i,j+1}$$

$$+ f_{i+1,j-1} - 2f_{i+1,j} + f_{i+1,j}]$$

- This then produces the following module

$$\frac{1}{\Delta x^2 \Delta y^2}$$



- We can also obtain this module as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\Delta x^2} \left[\begin{matrix} 1 & - & -2 & - & 1 \end{matrix} \right]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{\Delta y^2} \begin{matrix} 1 \\ | \\ -2 \\ | \\ 1 \end{matrix}$$

- Therefore

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{1}{\Delta x^2 \Delta y^2} \begin{matrix} 1 \\ | \\ -2 \\ | \\ 1 \end{matrix} \times \begin{matrix} 1 & - & -2 & - & 1 \end{matrix} = \frac{1}{\Delta x^2 \Delta y^2} \begin{matrix} 1 & - & -2 & - & 1 \\ | & & | & & | \\ -2 & - & 4 & - & -2 \\ | & & | & & | \\ 1 & - & -2 & - & 1 \end{matrix}$$

- To obtain $\nabla^4 f$, we must add up modules:

$$\nabla^4 f = \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}$$

- If $\Delta x = \Delta y = h$ then: