TOPIC 3 - THE ENERGY EQUATION AND CRITICAL FLOW

3.1 SPECIFIC ENERGY AND ALTERNATE DEPTHS OF FLOW IN WIDE CHANNELS

The Energy Equation and Open Channel Flow

- In our analysis of open channel flow, we *typically* make the following assumptions:
  
  - The streamlines are parallel.
  - Bed slope $S_0$ is small so that $\cos \theta_0 \approx 1$.
  - The kinetic energy correction factor is approximately equal to 1.0

- The first two assumptions allow us to assume that the pressure distribution in the vertical direction is hydrostatic.
The Bernoulli or mechanical energy equation for this case between any two points in a channel becomes

\[ d_1 + z_1 + \frac{\bar{u}_1^2}{2g} = d_2 + z_2 + \frac{\bar{u}_2^2}{2g} + h_{L_{1-2}} \]  \hspace{1cm} (3.1.1)

where

\[ z = \text{elevation from a datum to the channel bottom} \]  \hspace{1cm} (3.1.2)

\[ d = \text{fluid depth} \]  \hspace{1cm} (3.1.3)

\[ \bar{u} = \text{depth or cross sectional average velocity in the channel} \]  \hspace{1cm} (3.1.4)

\[ h_{L_{1-2}} = \text{total head loss in the channel between points 1 and 2, including friction losses and any losses due to obstructions} \]  \hspace{1cm} (3.1.5)
Specific Energy and Alternate Depths of Flow

The Specific Energy Diagram

• Let us consider wide rectangular channels.

• Define specific energy at a given cross section:

\[ E = d + \frac{\bar{u}^2}{2g} \]  

(3.1.6)

• If \( q \) is defined as the flow per unit width of a wide rectangular channel of width \( b \):

\[ \bar{u} = \frac{Q}{A} = \frac{qb}{db} = \frac{q}{d} \]  

(3.1.7)

• Thus specific energy becomes

\[ E = d + \frac{q^2}{2gd^2} \]  

(3.1.8)

• Solving for \( q \):

\[ q = d\sqrt{2g(E-d)} \]  

(3.1.9)
• For a fixed flow rate $q$, this equation represents the types of curves shown:

![Diagram showing types of curves]

• We note that for every specific energy $E$ and flow rate $q$, there are at most 2 associated flow depths $d$.

• Thus, for a given specific energy $E$ and flow rate $q$, we can have two alternate depths or stages:
  • An upper stage $d_u$ occurring with an associated tranquil flow, $\bar{u}_u$. This flow is referred to as **subcritical**.
  • A lower stage $d_l$ occurring with an associated rapid flow $\bar{u}_l$. This flow is referred to as **supercritical**.
• Note that both flows have the same specific energy and flow rate. Thus:

\[ E = d_u + \frac{\tilde{u}_u^2}{2g} = d_l + \frac{\tilde{u}_l^2}{2g} \]  

(3.1.10)

\[ q = \bar{u}_u d_u = \bar{u}_l d_l \]  

(3.1.11)
**Critical Depth**

- We can find the point of minimum energy on each specific energy curve for a given \( q \).

\[
\frac{\partial E}{\partial d} = 1 - \frac{q^2}{gd^3} = 0 \tag{3.1.12}
\]

- Solving, we find the depth at which the minimum specific energy occurs. This depth is called the *critical depth*, \( d_c \).

\[
gd^3 \frac{q}{c^2} = \frac{q}{g} \tag{3.1.13}
\]

\[
d_c = \left(\frac{q}{g}\right)^{1/3} \tag{3.1.14}
\]

- The critical depth is the flow depth for a given flow rate \( q \) for which specific energy is minimum.
• The critical flow depth delineates the two possible alternate flow depths:
  
  • \( d > d_c \): The flow is subcritical.
  
  • \( d < d_c \): The flow is supercritical.
• Re-writing equation (3.1.14) we have:

\[ q = \sqrt{g d_c^3} \]  \hspace{1cm} (3.1.15)

• However, noting that \( q = \bar{u}d \), we have

\[ \bar{u}_c = \frac{\sqrt{g d_c^3}}{d_c} \]  \hspace{1cm} (3.1.16)

• Thus, at the critical depth, the critical velocity equals

\[ \bar{u}_c = \sqrt{g d_c} \]  \hspace{1cm} (3.1.17)

• However, if we explore wave theory, we would find that a wave in a channel propagates at

\[ c = \sqrt{gd} \]  \hspace{1cm} (3.1.18)

• The wave speed \( c \) is called the wave celerity.
• It is the speed at which a long wave or shallow water wave propagates. A long or shallow water wave has a wavelength that is longer than the depth.
• Thus from equation (3.1.17) we note that

\[
\frac{\tilde{u}_c}{\sqrt{gd_c}} = 1
\]  

(3.1.19)

• Thus, for critical flow, we note that the water speed is equal to the speed at which a wave propagates.

  • This is very important in hydraulic theory since this delineates whether or not disturbances can propagate upstream.

• However, equation (3.1.19) defines the Froude number as we noted in topic 1.6. Thus

\[
F_R = \frac{\tilde{u}_c}{\sqrt{gd_c}} = 1
\]  

(3.1.20)

• Recall that Froude number also expresses the ratio of convective inertial forces (also kinetic energy) to gravity forces.

• In general, Froude number indicates whether the flow is sub- or supercritical.

  • $F_R < 1$: Flow is subcritical and disturbances can propagate upstream.
  
  • $F_R > 1$: Flow is supercritical and disturbances can not propagate upstream.
**Summary of Critical Depth**

- A flow condition, i.e. a certain rate of discharge flowing at a certain depth, is completely specified by any two of the variables \( d, q, \bar{u} \) and \( E \) except the combination \( q \) and \( E \) which yields in general two alternate stages of flow.

- For any value of \( E \) there exists a critical depth, \( d_c \), for which flow is a maximum.

  • Consider

\[
E = \frac{\bar{u}^2}{2g} + d
\]  

(3.1.21)

• At critical flow we have

\[
\bar{u}_c = \sqrt{gd_c}
\]  

(3.1.22)

• At the critical depth we showed that the specific energy was a minimum:

\[
E_{min} = \frac{gd_c}{2g} + d_c
\]  

(3.1.23)

\[
E_{min} = \frac{3}{2}d_c
\]  

(3.1.24)
• Thus, the depth at which flow is maximum for a given specific energy:

\[ d_c = \frac{2}{3} E_{min} \]  

(3.1.25)

• See the figure to help visualize that for a given \( E \), the critical depth will deliver more flow than any other depth.

• For any value of \( q \), there exists a critical depth for which the specific energy is a minimum.

\[ d_c = \left( \frac{q}{g} \right)^{1/3} \]  

(3.1.26)

• When the flow occurs at the critical depth:
  
  • The depth-specific energy relationship is

\[ d_c = \frac{2}{3} E \]  

(3.1.27)
• The velocity head is half the depth:

\[
\frac{\bar{u}_c^2}{2g} = \left( \frac{q}{d_c} \right)^2 \cdot \frac{1}{2g}
\]  
(1.2.28)

\[
\frac{\bar{u}_c^2}{2g} = \frac{q^2}{g} \cdot \frac{1}{2d^2_c}
\]  
(1.2.29)

\[
\frac{\bar{u}_c^2}{2g} = d_c^3 \cdot \frac{1}{2d^2_c}
\]  
(1.2.30)

\[
\frac{\bar{u}_c}{2g} = \frac{d_c}{2}
\]  
(1.2.31)

• For any flow condition other than critical, there exists an alternate stage at which the same rate of discharge, \( q \), is carried by the same specific energy, \( E \). The alternate depths are found by solving:

\[
E = d + \frac{q^2}{2gd^2}
\]  
(3.1.32)
• Alternatively, we can use plots of $E$ versus $d$ for given values of $q$.
• Or use plots of $q$ versus $d$ for constant values of $E$
Channel Slope and Alternate Depths of Flow

• Recall that for steady uniform flow, the depth of flow determined only by the rate of discharge, the shape and roughness of the cross section and the slope of the stream bed:

\[ Q = \frac{\kappa}{n} A R_H^{2/3} S_0^{1/2} \]  \hspace{1cm} (3.1.33)

where \( \kappa \) = units coefficient  \hspace{1cm} (3.1.34)

\( A \) = cross sectional area of water  \hspace{1cm} (3.1.35)

\( n \) = Manning coefficient  \hspace{1cm} (3.1.36)

\( R_H \) = hydraulic radius  \hspace{1cm} (3.1.37)

\( S_0 \) = bed slope  \hspace{1cm} (3.1.38)

• For a rectangular channel

\[ R_H = \frac{d_w}{2d + w} \]  \hspace{1cm} (3.1.39)
• If the channel is wide, $w \gg d$, then

$$R_H \equiv \frac{dw}{d} = d \quad (3.1.40)$$

• Thus

$$q = \frac{Q}{w} \quad (1.2.41)$$

$$q = \frac{\kappa wdd^{2/3}S_0^{1/2}}{n} \quad (1.2.42)$$

$$q = \frac{\kappa d^{5/3}S_0^{1/2}}{n} \quad (1.2.43)$$

• Thus

$$d_0 = \left( \frac{qn}{\kappa S_0^{1/2}} \right)^{3/5} \quad (3.1.44)$$
• The normal flow depth is determined by the discharge per unit width, $q$; the roughness, $n$; the bottom slope, $S_0$.

• The actual flow in a channel is critical if the normal depth, $d_0$, is equal to the critical depth, $d_c$.

• Thus we can conveniently divide channel flow into one of three regimes depending on the relationship between $d_c$ and $d_0$.

  • $d_0 = d_c$  Critical flow
  • $d_0 > d_c$  Subcritical flow
  • $d_0 < d_c$  Supercritical flow
• Alternatively, we can look at the Froude number to classify the flow:
  
  • $F_R = 1$ Critical flow
  • $F_R < 1$ Subcritical flow
  • $F_R > 1$ Supercritical flow

  • Again we can look at the Froude number to help determine how disturbances propagate in the channel.
    
    • If a wave travels upstream and downstream, then the flow is subcritical and $F_R < 1$.
    
    • If a wave is completely washed downstream by the fluid velocity, then the flow is supercritical and $F_R > 1$.

• Yet another way to consider sub- and supercritical flow is to examine the slope that produces a normal depth equal to the critical depth, $S_c$, and compare this to bottom slope, $S_0$.

• Thus the critical slope of a channel will just sustain the given rate of discharge in uniform flow at the critical depth.

$$S_c = \frac{q^2 n^2}{\kappa^2 d_c^{10/3}} \quad (3.1.45)$$
• We can classify flow according to the following:
  • $S_0 = S_c$  Critical flow and critical slope
  • $S_0 < S_c$  Subcritical flow and mild slope
  • $S_0 > S_c$  Supercritical flow and steep slope

Thus hydraulic steepness (mild, critical, steep) of a channel is determined by its channel slope and roughness and shape.
When flow is near critical, a small change in specific energy, bed roughness or bed slope produces relatively large changes in depth.

- Critical depth is a point of instability in the flow regime.
- With flow at or near critical depth, there will be an undulating or wavy free surface.
- *It is poor engineering practice to design channels at or near the critical depth or slope.*
- The only time we want to intentionally force critical flow is either for flow measurement devices such as weirs or for flow control of some type.