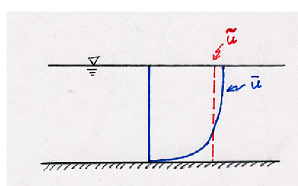
1.5 DEVELOPMENT OF THE DEPTH AVERAGED GOVERNING EQUATIONS

General Considerations

- We are developing a hierarchy of equations averaged over various scales.
    - Molecular Scale:
        - Navier-Stokes Equation + Continuity Equation
        - u, v, w, p
        - Valid for all flows down to the molecular scale
        - We can not resolve down to the viscous dissipation scale for normal computations
    - Turbulent Scale:
        - Reynolds Equation + Continuity Equation
        - $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$ and $\overline{u'u'}$, $\overline{u'v'}$, $\overline{u'w'}$, $\overline{v'v'}$, $\overline{v'w'}$, $\overline{w'w'}$
        - Need to select a turbulence closure model (i.e. a set of ***constitutive*** relationships)
        - Valid for all flows down to the turbulent averaging scale (in time T and related space scales)

- Let us examine spatial averaging over the depth dimension $\rightarrow$ ***depth averaged flow***
    - River and channel flows
    - Estuarine and coastal ocean flows



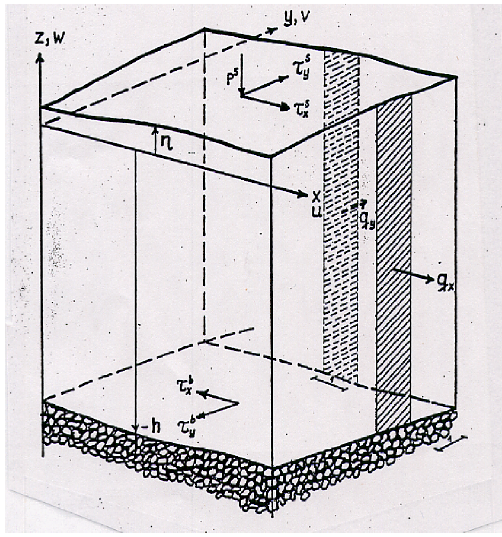
- A key assumption for depth averaging is that the ***flow in the vertical direction is small***.
- This implies that all terms in the z-direction Reynolds Equation are ***small*** compared to the gravity and pressure terms.
    - Thus the z-direction Reynolds Equation reduces to

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \qquad (1.5.1)$$

- This implies that the pressure distribution over the vertical is hydrostatic.

Depth Integrated Continuity Equation

- Consider the geoid to be defined at $z=0$, the free surface (water-air interface) at $z=\eta$, and the bottom (water-sediment interface) at $z=-h$.



- Depth averaged velocities are defined as:

$$\tilde{u} \equiv \frac{1}{H}\int_{-h}^{\eta} \bar{u}\,dz \qquad (1.5.2)$$

$$\tilde{v} \equiv \frac{1}{H}\int_{-h}^{\eta} \bar{v}\,dz \qquad (1.5.3)$$

- Where total water column height is defined as:

$$H = \eta + h \qquad (1.5.4)$$

- Flow rate over the vertical is defined as:

$$q_x \equiv \int_{-h}^{\eta} \bar{u}\,dz = \tilde{u}H \qquad (1.5.5)$$

$$q_y \equiv \int_{-h}^{\eta} \bar{v}\,dz = \tilde{v}H \qquad (1.5.6)$$

- Assuming incompressible turbulent time averaged flow, the appropriate form of the continuity equation is:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \quad (1.5.7)$$

- Now let's vertically average:

$$\frac{1}{H}\int_{-h}^{\eta} \frac{\partial \bar{u}}{\partial x} dz + \frac{1}{H}\int_{-h}^{\eta} \frac{\partial \bar{v}}{\partial y} dz + \frac{1}{H}\int_{-h}^{\eta} \frac{\partial \overline{w}}{\partial z} dz = 0 \quad (1.5.8)$$

- Multiplying through by H and evaluating the last integral:

$$\int_{-h(x,y)}^{\eta(x,y,t)} \frac{\partial \bar{u}}{\partial x} dz + \int_{-h(x,y)}^{\eta(x,y,t)} \frac{\partial \bar{v}}{\partial y} dz + \overline{w}(\eta) - \overline{w}(-h) = 0 \quad (1.5.9)$$

- Using Leibnitz's Rule, we know that:

$$\int_{A(x,y)}^{B(x,y,t)} \frac{\partial f}{\partial x} dz = \frac{\partial}{\partial x}\int_{A}^{B} f\, dz - f\big|_{z=B}\frac{\partial B}{\partial x} + f\big|_{z=A}\frac{\partial A}{\partial x} \quad (1.5.10)$$

- Substituting in for the remaining integral terms using Leibnitz's Rule and re-arranging:

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} \bar{u}\, dz + \frac{\partial}{\partial y}\int_{-h}^{\eta} \bar{v}\, dz - \left[\bar{u}\frac{\partial \eta}{\partial x} + \bar{v}\frac{\partial \eta}{\partial y} - \overline{w}\right]_{z=\eta} + \left[\bar{u}\frac{\partial(-h)}{\partial x} + \bar{v}\frac{\partial(-h)}{\partial y} - \overline{w}\right]_{z=-h} = 0 \quad (1.5.11)$$

- Velocities at the free surface and bottom may be expressed as:

$$\overline{w}\big|_{z=\eta} = \left.\frac{D\eta}{Dt}\right|_{z=\eta} = \frac{\partial \eta}{\partial t} + \bar{u}\big|_{z=\eta}\frac{\partial \eta}{\partial x} + \bar{v}\big|_{z=\eta}\frac{\partial \eta}{\partial y} \quad (1.5.12)$$

$$\overline{w}\big|_{z=-h} = \left.\frac{D(-h)}{Dt}\right|_{z=-h} = \frac{\partial(-h)}{\partial t} + \bar{u}\big|_{z=-h}\frac{\partial(-h)}{\partial x} + \bar{v}\big|_{z=-h}\frac{\partial(-h)}{\partial y} \quad (1.5.13)$$

- Re-writing above and noting that $\frac{\partial h(x,y)}{\partial t} \equiv 0$

$$-\left[\bar{u}\frac{\partial \eta}{\partial x} + \bar{v}\frac{\partial \eta}{\partial y} - \overline{w}\right]_{z=\eta} = \frac{\partial \eta}{\partial t} \quad (1.5.14)$$

$$\left[\bar{u}\frac{\partial(-h)}{\partial x} + \bar{v}\frac{\partial(-h)}{\partial y} - \overline{w}\right]_{z=-h} = 0 \quad (1.5.15)$$

- Substituting these expressions into the continuity equation we have:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\int_{-h}^{\eta} \bar{u}\,dz + \frac{\partial}{\partial y}\int_{-h}^{\eta} \bar{v}\,dz = 0 \qquad (1.5.16)$$

- However by definition we have:

$$\int_{-h}^{\eta} \bar{u}\,dz \equiv q_x \qquad (1.5.17)$$

$$\int_{-h}^{\eta} \bar{v}\,dz \equiv q_y \qquad (1.5.18)$$

- Again substituting:

$$\frac{\partial \eta}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \qquad (1.5.19)$$

- Noting that $q_x = H\tilde{u}$, $q_y = H\tilde{v}$ where $\tilde{u}, \tilde{v}$ equal the vertically averaged velocities:

$$\frac{\partial \eta}{\partial t} + \frac{\partial(\tilde{u}H)}{\partial x} + \frac{\partial(\tilde{v}H)}{\partial y} = 0 \qquad (1.5.20)$$

Depth Integrated Reynolds Equations

- Consider the *x*-direction Reynolds equation:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau_{xx}^{t/m}}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau_{yx}^{t/m}}{\partial y} + \frac{1}{\rho_0}\frac{\partial \tau_{zx}^{t/m}}{\partial z} \qquad (1.5.21)$$

$$\text{where } \tau_{xx}^{t/m} = \mu\frac{\partial \bar{u}}{\partial x} - \rho_0\overline{u'u'} \qquad (1.5.22)$$

$$\tau_{yx}^{t/m} = \mu\frac{\partial \bar{u}}{\partial y} - \rho_0\overline{u'v'} \qquad (1.5.23)$$

$$\tau_{zx}^{t/m} = \mu\frac{\partial \bar{u}}{\partial z} - \rho_0\overline{u'w'} \qquad (1.5.24)$$

- Now consider the hydrostatic pressure equation:

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho}g \qquad (1.5.25)$$

- Integrating this equation between the free surface at $z=\eta$ and some level z

$$\int_{p_s(x,y)}^{\bar{p}(x,y,z)} \partial \bar{p} = -\int_{z=\eta}^{z} \bar{\rho} g \partial z \tag{1.5.26}$$

where p_s = pressure at the free surface (1.5.27)

- Assuming that density is constant:

$$\bar{p} - p_s = -(\rho g z - \rho g \eta) \tag{1.5.28}$$

$$\bar{p} = p_s + \rho g \eta - \rho g z \tag{1.5.29}$$

- The pressure gradient term in the x-direction Reynolds equation becomes

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = -\frac{1}{\rho}\frac{\partial p_s}{\partial x} - g\frac{\partial \eta}{\partial x} \tag{1.5.30}$$

- Assuming that surface pressure does not vary spatially:

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = -g\frac{\partial \eta}{\partial x} \tag{1.5.31}$$

- Thus the x-direction Reynolds equation with the assumptions of constant density flow, and hydrostatic pressure distribution and constant atmospheric pressure becomes:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -g\frac{\partial \eta}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xx}^{t/m}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}^{t/m}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zx}^{t/m}}{\partial z} \tag{1.5.32}$$

- Now add $\bar{u}$ times the continuity equation to the above equation:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \bar{v}}{\partial y} + \bar{u}\frac{\partial \bar{w}}{\partial z} =$$

$$-g\frac{\partial \eta}{\partial x} + \frac{1}{\rho}\left\{\frac{\partial \tau_{xx}^{t/m}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}^{t/m}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zx}^{t/m}}{\partial z}\right\} \tag{1.5.33}$$

- Re-arranging

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial (\bar{u}\bar{v})}{\partial y} + \frac{\partial (\bar{u}\bar{w})}{\partial z} =$$

$$-g\frac{\partial \eta}{\partial x} + \frac{1}{\rho}\left\{\frac{\partial \tau_{xx}^{t/m}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}^{t/m}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zx}^{t/m}}{\partial z}\right\} \tag{1.5.34}$$

• Vertically averaging these equations:

$$\int_{-h}^{\eta} \frac{\partial \bar{u}}{\partial t} dz + \int_{-h}^{\eta} \frac{\partial \bar{u}^2}{\partial x} dz + \int_{-h}^{\eta} \frac{\partial (\bar{u}\bar{v})}{\partial y} dz + \int_{-h}^{\eta} \frac{\partial (\bar{u}\bar{w})}{\partial z} dz =$$

$$-g \int_{-h}^{\eta} \frac{\partial \eta}{\partial x} dz + \int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial \tau_{xx}^{t/m}}{\partial x} dz + \int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial \tau_{yx}^{t/m}}{\partial y} dz + \int_{-h}^{\eta} \frac{1}{\rho} \frac{\partial \tau_{zx}^{t/m}}{\partial z} dz \qquad (1.5.35)$$

• Apply Leibnitz's rule as follows:

$$\int_{-h(x,y)}^{\eta(x,y,t)} \frac{\partial f}{\partial t} dz = \frac{\partial}{\partial t} \int_{-h}^{\eta} f\, dz - f\big|_{z=\eta} \frac{\partial \eta}{\partial t} + f\big|_{z=-h} \frac{\partial (-h)}{\partial t} \qquad (1.5.36)$$

$$\text{where } f = \bar{u},\ \bar{u}^2,\ \bar{u}\bar{v},\ \eta,\ \tau_{xx},\ \tau_{yx} \qquad (1.5.37)$$

• Furthermore, we note that:

$$\int_{-h}^{\eta} \frac{\partial g}{\partial z} dz = \int_{-h}^{\eta} \partial g = g\big|_{z=\eta} - g\big|_{z=-h} \qquad (1.5.38)$$

$$\text{where } g = \bar{u}\bar{w},\ \tau_{zx} \qquad (1.5.39)$$

• Substituting and re-arranging we have:

$$\frac{\partial}{\partial t} \int_{-h}^{\eta} \bar{u}\, dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} \bar{u}^2 dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} \bar{u}\bar{v}\, dz$$

$$-\left[\bar{u}\left(\frac{\partial \eta}{\partial t} + \bar{u}\frac{\partial \eta}{\partial x} + \bar{v}\frac{\partial \eta}{\partial y} - \bar{w}\right)\right]_{z=\eta} + \left[\bar{u}\frac{\partial (-h)}{\partial t} + \bar{u}\frac{\partial (-h)}{\partial x} + \bar{v}\frac{\partial (-h)}{\partial y} - \bar{w}\right]_{z=-h} =$$

$$-g\frac{\partial}{\partial x} \int_{-h}^{\eta} \eta\, dz + g\eta \frac{\partial \eta}{\partial x} - g\eta \frac{\partial (-h)}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x} \int_{-h}^{\eta} \tau_{xx}^{t/m} dz$$

$$+\frac{1}{\rho}\frac{\partial}{\partial y} \int_{-h}^{\eta} \tau_{yx}^{t/m} dz - \frac{1}{\rho}\left[\tau_{xx}^{t/m}\frac{\partial \eta}{\partial x} + \tau_{yx}^{t/m}\frac{\partial \eta}{\partial y} - \tau_{zx}^{t/m}\right]_{z=\eta}$$

$$\frac{1}{\rho}\left[\tau_{xx}^{t/m}\frac{\partial (-h)}{\partial x} + \tau_{yx}^{t/m}\frac{\partial (-h)}{\partial y} - \tau_{zx}^{t/m}\right]_{z=-h} \qquad (1.5.40)$$

• However as we noted before:

$$\left[\frac{\partial\eta}{\partial t} + \bar{u}\frac{\partial\eta}{\partial x} + \bar{v}\frac{\partial\eta}{\partial y} - \bar{w}\right]_{z=\eta} = 0 \qquad (1.5.41)$$

$$\left[\frac{\partial(-h)}{\partial t} + \bar{u}\frac{\partial(-h)}{\partial x} + \bar{v}\frac{\partial(-h)}{\partial y} - \bar{w}\right]_{z=-h} = 0 \qquad (1.5.42)$$

• By performing a stress balance at the surface, it can be shown that τ_x^s = applied surface stress in the x-direction ***and*** parallel to the surface.

$$\tau_x^s = \left[-\tau_{xx}^{\ \ t/m}\frac{\partial\eta}{\partial x} + \tau_{yx}^{\ \ t/m}\frac{\partial\eta}{\partial y} - \tau_{zx}^{\ \ t/m}\right]_{z=\eta} \qquad (1.5.43)$$

• Similarly at the bottom:

$$\tau_x^b = -\left[\tau_{xx}^{\ \ t/m}\frac{\partial(-h)}{\partial x} + \tau_{yx}^{\ \ t/m}\frac{\partial(-h)}{\partial y} - \tau_{zx}^{\ \ t/m}\right]_{z=-h} \qquad (1.5.44)$$

• Substituting reduces the x-momentum equation to:

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\bar{u}\,dz + \frac{\partial}{\partial x}\int_{-h}^{\eta}\bar{u}^2\,dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}\bar{u}\bar{v}\,dz =$$

$$-g\frac{\partial}{\partial x}\int_{-h}^{\eta}\eta\,dz + g\eta\frac{\partial\eta}{\partial x} - g\eta\frac{\partial(-h)}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\eta}\tau_{xx}^{\ \ t/m}dz + \frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\eta}\tau_{yx}^{\ \ t/m}dz + \frac{\tau_x^s}{\rho} - \frac{\tau_x^b}{\rho} \qquad (1.5.45)$$

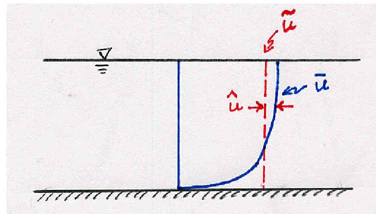
• Let us define the depth averaged variable as

$$\tilde{\alpha} \equiv \frac{1}{H}\int_{-h}^{\eta}\bar{\alpha}\,dz \qquad (1.5.46)$$

• The Reynolds averaged quantity is then defined as the sum of the depth averaged variable and the deviation from the depth averaged variable

$$\bar{\alpha} \equiv \tilde{\alpha} + \hat{\alpha} \qquad (1.5.47)$$

- Thus we define velocities in terms of the depth averaged quantity and the deviation from the depth averaged quantity



- Thus spatial averaging is applied as:

$$\int_{-h}^{\eta} \bar{u}\, dz \equiv H\tilde{u} \tag{1.5.48}$$

$$\int_{-h}^{\eta} \bar{v}\, dz \equiv H\tilde{u} \tag{1.5.49}$$

- Furthermore we let:

$$\bar{u}(x, y, z, t) = \tilde{u}(x, y, t) + \hat{u}(x, y, z, t) \tag{1.5.50}$$

$$\bar{v}(x, y, z, t) = \tilde{v}(x, y, t) + \hat{v}(x, y, z, t) \tag{1.5.51}$$

- This implies that:

$$\int_{-h}^{\eta} \hat{u}\, dz = 0 \text{ and } \int_{-h}^{\eta} \hat{v}\, dz = 0 \tag{1.5.52}$$

- Hence

$$\int_{-h}^{\eta} \bar{u}^2 dz = \int_{-h}^{\eta} (\tilde{u}^2 + \partial\tilde{u}\hat{u} + \hat{u}^2) dz \tag{1.5.53}$$

$$\int_{-h}^{\eta} \bar{u}^2 dz = \tilde{u}^2 \int_{-h}^{\eta} dz + 2\tilde{u} \int_{-h}^{\eta} \hat{u}\, dz + \int_{-h}^{\eta} \hat{u}^2 dz \tag{1.5.54}$$

$$\int_{-h}^{\eta} \bar{u}^2 dz = H\tilde{u}^2 + \int_{-h}^{\eta} \hat{u}^2 dz \tag{1.5.55}$$

- Similarly

$$\int_{-h}^{\eta} \bar{u}\bar{v}\, dz = \int_{-h}^{\eta} (\tilde{u}\tilde{v} + \tilde{u}\hat{v} + \hat{u}\tilde{v} + \hat{u}\hat{v}) dz = \tilde{u}\tilde{v}H + \int_{-h}^{\eta} \hat{u}\hat{v}\, dz \tag{1.5.56}$$

- Finally

$$\int_{-h}^{\eta} \eta dz = \eta \int_{-h}^{\eta} dz = \eta H \tag{1.5.57}$$

- Substituting and re-arranging:

$$\frac{\partial(H\tilde{u})}{\partial t} + \frac{\partial(H\tilde{u}^2)}{\partial x} + \frac{\partial(H\tilde{u}\tilde{v})}{\partial y} =$$

$$-g\frac{\partial(\eta H)}{\partial x} + g\eta\frac{\partial \eta}{\partial x} + g\eta\frac{\partial h}{\partial x} + \frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\frac{\tau_{xx}^{t/m}}{\rho} - \hat{u}^2\right)dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}\left(\frac{\tau_{yx}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz + \frac{\tau_x^s}{\rho} - \frac{\tau_x^b}{\rho} \tag{1.5.58}$$

- Expanding the terms involving gravity:

$$-g\frac{\partial(\eta H)}{\partial x} + g\eta\frac{\partial \eta}{\partial x} + g\eta\frac{\partial h}{\partial x} = g\left[-\eta\frac{\partial H}{\partial x} - H\frac{\partial \eta}{\partial x} + \eta\frac{\partial(\eta + h)}{\partial x}\right] \tag{1.5.59}$$

$$-g\frac{\partial(\eta H)}{\partial x} + g\eta\frac{\partial \eta}{\partial x} + g\eta\frac{\partial h}{\partial x} = -gH\frac{\partial \eta}{\partial x} \tag{1.5.60}$$

- We also note that the acceleration terms can be expanded as:

$$\frac{\partial(H\tilde{u})}{\partial t} = \tilde{u}\frac{\partial H}{\partial t} + H\frac{\partial \tilde{u}}{\partial t} \tag{1.5.61}$$

$$\frac{\partial(H\tilde{u})}{\partial t} = \tilde{u}\frac{\partial(\eta + h)}{\partial t} + H\frac{\partial \tilde{u}}{\partial t} \tag{1.5.62}$$

$$\frac{\partial(H\tilde{u})}{\partial t} = \tilde{u}\frac{\partial \eta}{\partial t} + H\frac{\partial \tilde{u}}{\partial t} \tag{1.5.63}$$

$$\frac{\partial(H\tilde{u}^2)}{\partial x} = \tilde{u}\frac{\partial(H\tilde{u})}{\partial x} + H\tilde{u}\frac{\partial \tilde{u}}{\partial x} \tag{1.5.64}$$

$$\frac{\partial(H\tilde{u}\tilde{v})}{\partial y} = \tilde{u}\frac{\partial(H\tilde{v})}{\partial y} + H\tilde{v}\frac{\partial \tilde{u}}{\partial y} \tag{1.5.65}$$

- Substituting in the gravity and acceleration term re-arrangements:

$$H\frac{\partial\tilde{u}}{\partial t} + H\tilde{u}\frac{\partial\tilde{u}}{\partial x} + H\tilde{v}\frac{\partial\tilde{u}}{\partial y} + \tilde{u}\left[\frac{\partial\eta}{\partial t} + \frac{\partial(H\tilde{u})}{\partial x} + \frac{\partial(H\tilde{v})}{\partial y}\right]$$

$$= -gH\frac{\partial\eta}{\partial x} + \frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\frac{\tau_{xx}^{t/m}}{\rho} - \hat{u}^2\right)dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}\left(\frac{\tau_{yx}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz + \frac{\tau_x^s}{\rho} - \frac{\tau_x^b}{\rho} \qquad (1.5.66)$$

- It is clear that the depth averaged continuity equation is embedded in the previous equation and therefore drops out. Dividing through by H results in the depth averaged conservation of momentum equation in non-conservative form (refers to us having altered the form of the acceleration terms).

$$\frac{\partial\tilde{u}}{\partial t} + \tilde{u}\frac{\partial\tilde{u}}{\partial x} + \tilde{v}\frac{\partial\tilde{u}}{\partial y} = -g\frac{\partial\eta}{\partial x} +$$

$$\frac{1}{H}\frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\frac{\tau_{xx}^{t/m}}{\rho} - \hat{u}^2\right)dz + \frac{1}{H}\frac{\partial}{\partial y}\int_{-h}^{\eta}\left(\frac{\tau_{yx}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz + \frac{1}{H}\frac{\tau_x^s}{\rho} - \frac{1}{H}\frac{\tau_x^b}{\rho} \qquad (1.5.67)$$

- The ***Shallow Water Equations*** are collectively written as:

$$\frac{\partial\eta}{\partial t} + \frac{\partial(\tilde{u}H)}{\partial x} + \frac{\partial(\tilde{v}H)}{\partial y} = 0 \qquad (1.5.68)$$

$$\frac{\partial\tilde{u}}{\partial t} + \tilde{u}\frac{\partial\tilde{u}}{\partial x} + \tilde{v}\frac{\partial\tilde{u}}{\partial y} = -g\frac{\partial\eta}{\partial x} +$$

$$\frac{1}{H}\frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\frac{\tau_{xx}^{t/m}}{\rho} - \hat{u}^2\right)dz + \frac{1}{H}\frac{\partial}{\partial y}\int_{-h}^{\eta}\left(\frac{\tau_{yx}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz + \frac{1}{H}\frac{\tau_x^s}{\rho} - \frac{1}{H}\frac{\tau_x^b}{\rho} \qquad (1.5.69)$$

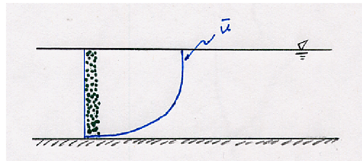
$$\frac{\partial\tilde{v}}{\partial t} + \tilde{u}\frac{\partial\tilde{v}}{\partial x} + \tilde{v}\frac{\partial\tilde{v}}{\partial y} = -g\frac{\partial\eta}{\partial y} +$$

$$\frac{1}{H}\frac{\partial}{\partial x}\int_{-h}^{\eta}\left(\frac{\tau_{xy}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz + \frac{1}{H}\frac{\partial}{\partial y}\int_{-h}^{\eta}\left(\frac{\tau_{yy}^{t/m}}{\rho} - \hat{v}^2\right)dz + \frac{1}{H}\frac{\tau_y^s}{\rho} - \frac{1}{H}\frac{\tau_y^b}{\rho} \qquad (1.5.70)$$

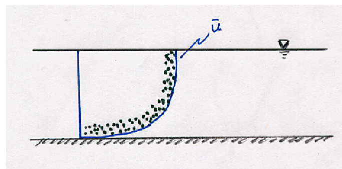
- The Shallow Water Equations were established in 1775 by Laplace.
- The Momentum Conservation Statements are quite similar to the Reynolds equations with the following exceptions:
  - Variables are now depth averaged quantities.
  - The z-dimension has been eliminated.
  - There are convective inertia forces caused by the flow deviation from the depth averaged velocities $\tilde{u}, \tilde{v}$.
- These equations have built into them 3 levels of averaging:
  - Averaging over the molecular time/space scale
  - Averaging over the turbulent time/space scale
  - Averaging over the depth space scale
  - The latter two produce momentum transport terms that are intimately related to the convective terms.

- There are now three mechanisms of momentum transfer built into these equations:
  - $\int_{-h}^{\eta} \nu \frac{\partial \bar{u}}{\partial x} dz$ type terms are the ***viscous stresses*** and represent the averaged effect of molecular motions. These terms are necessary since we do not directly simulate the momentum transfer via molecular level collisions.
  - $\int_{-h}^{\eta} \overline{u'u'} dz$ type terms are the ***turbulent Reynolds stresses*** and represent the averaged effect of momentum transfer due to turbulent fluctuations. These terms are necessary since we are not directly simulating momentum transfer via turbulent fluctuations.
  - $\int_{-h}^{\eta} \hat{u}\hat{u} dz$ type terms represent the spreading of momentum over the water column. This process is known as ***momentum dispersion***. These terms are necessary since we are no longer directly simulating this process via the actual depth varying velocity profiles. They spread momentum laterally.

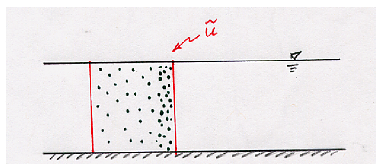
- Momentum dispersion can be conceptualized by following a pollutant spread over the water column.
  - Non-depth average simulation at $t = t_0$



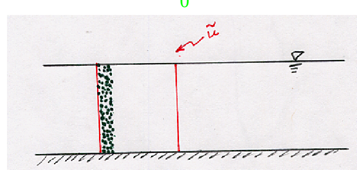
  - Non-depth average simulation at $t = t_1$



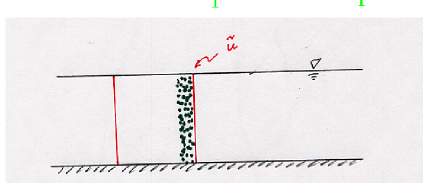
  - Depth averaging the result of the non-depth averaged simulation at $t = t_1$



  - Depth averaged simulation at $t = t_0$



  - Depth averaged simulation at $t = t_1$ without dispersion terms



- Thus, dispersion terms are necessary!

- The Shallow Water equations greatly simplify flow computations in free surface water bodies.
  - Reduce the number of p.d.e.'s from 4 to 3.
  - Reduce the complexity of the variables

$$\tilde{u}(x, y, t), \tilde{v}(x, y, t), \eta(x, y, t) \text{ instead of} \tag{1.5.71}$$

$$\bar{u}(x, y, z, t), \bar{v}(x, y, z, t), \bar{w}(x, y, z, t), \bar{p}(x, y, z, t) \tag{1.5.72}$$

  - Built-in positioning of the free surface boundary which is typically unknown when applying the Reynolds equation.
- The Shallow Water equations include 10 additional unknowns as compared to the Navier-Stokes equations:
  - $(\widetilde{\overline{u'u'}}), (\widetilde{\overline{u'v'}}), (\widetilde{\overline{v'v'}}) \rightarrow$ Lateral turbulent momentum diffusion
  - $\widetilde{\hat{u}\hat{u}}, \widetilde{\hat{u}\hat{v}}, \widetilde{\hat{v}\hat{v}} \rightarrow$ Lateral momentum dispersion related to vertical velocity profile
  - $\tau_x^s, \tau_y^s \rightarrow$ Applied free surface stress
  - $\tau_x^b, \tau_y^b \rightarrow$ Applied bottom stress. It is related to the vertical velocity profile, momentum transport through the water column, bottom roughness.

- These 10 additional unknowns require that 10 ***constitutive relationships*** are provided in order to ***close*** the system.
- A very simple closure model for the combined lateral momentum diffusion (due to turbulence) and dispersion (due to averaging out vertical velocity profile) is:

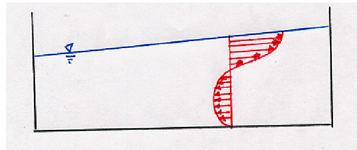
$$\int_{-h}^{\eta}\left(\frac{\tau_{xx}^{t/m}}{\rho} - \hat{u}^2\right)dz = E_{xx}\frac{\partial(H\tilde{u})}{\partial x} \tag{1.5.73}$$

$$\int_{-h}^{\eta}\left(\frac{\tau_{yy}^{t/m}}{\rho} - \hat{v}^2\right)dz = E_{yy}\frac{\partial(H\tilde{v})}{\partial y} \tag{1.5.74}$$

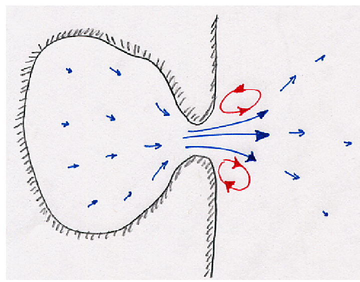
$$\int_{-h}^{\eta}\left(\frac{\tau_{xy}^{t/m}}{\rho} - \hat{u}\hat{v}\right)dz = E_{xy}\left[\frac{\partial(H\tilde{u})}{\partial y} + \frac{\partial(H\tilde{v})}{\partial x}\right] \tag{1.5.75}$$

  - E_{xx}, E_{yy} and E_{xy} are called the ***eddy dispersion*** coefficients.
  - This model assumes that the dispersion process dominates the turbulent momentum diffusion process which dominates the molecular momentum diffusion process.
  - In a typical gravity-driven open channel flow, the lateral momentum dispersion terms do not play a major role in the momentum balance equations - they can be neglected.

- In cases where a strong vertical velocity profile exists (i.e. wind driven flow in deep water), the lateral dispersion terms may be important due to the $\hat{u}, \hat{v}$ contributions. The importance of these terms is tied to the advective terms.



- In cases where very strong lateral gradients in the horizontal velocity profile exist (e.g. strong flow emerging into a receiving water body), the lateral dispersion terms may be important due to the u', v' contributions. The importance of these terms is related to the convective terms.



- The eddy dispersion model provides equations for 6 unknowns (since unknowns were consolidated). We still need constitutive relationships for 4 unknowns.

- Surface stress is very small unless wind is blowing over the water. Wind stress relationships depend on sea surface roughness and 10 m wind speed. They are not related to the water speed.

- Bottom stress is closed by empirical relationships:

$$\frac{\tau_x^b}{\rho} = c_f(\tilde{u}^2 + \tilde{v}^2)^{1/2}\tilde{u} \quad (1.5.76)$$

$$\frac{\tau_y^b}{\rho} = c_f(\tilde{u}^2 + \tilde{v}^2)^{1/2}\tilde{v} \quad (1.5.77)$$

where

$$c_f = \text{friction factor} \quad (1.5.78)$$

$$c_f = \frac{1}{8}f_{DW} \rightarrow \text{Darcy Weisbach} \quad (1.5.79)$$

$$c_f = \frac{g}{c^2} \rightarrow \text{Chezy} \quad (1.5.80)$$

$$c_f = \frac{n^2 g}{h^{1/3}} \rightarrow \text{Manning} \quad (1.5.81)$$