2.4 VELOCITY DISTRIBUTIONS IN OPEN CHANNELS

• In our analysis of steady uniform flow we neglected the effects of the depth and/or laterally varying velocity distribution.

Derivation of Velocity Distribution in a Wide Open Channel with Steady Uniform Flow

Governing Equations

• Instead of assuming depth averaged flow, we will derive a more detailed expression for flow and examine flow structure over the vertical.
  
  • Thus we average our equations over turbulent time/space scale and no longer average over the vertical.

• Thus we must apply the Reynolds’ equations.

• We examine steady uniform flow in a wide channel:

\[
\begin{align*}
\frac{D \bar{u}}{Dt} &= g \sin \theta_0 - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\mu \frac{\partial \bar{u}}{\partial x}}{\rho} - u' \bar{u}' \right) + \frac{\partial}{\partial y} \left( \frac{\mu \frac{\partial \bar{u}}{\partial y}}{\rho} - u' \bar{v}' \right) + \frac{\partial}{\partial z} \left( \frac{\mu \frac{\partial \bar{u}}{\partial z}}{\rho} - u' \bar{w}' \right) \\
\frac{D \bar{v}}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\mu \frac{\partial \bar{v}}{\partial x}}{\rho} - v' \bar{v}' \right) + \frac{\partial}{\partial y} \left( \frac{\mu \frac{\partial \bar{v}}{\partial y}}{\rho} - v' \bar{v}' \right) + \frac{\partial}{\partial z} \left( \frac{\mu \frac{\partial \bar{v}}{\partial z}}{\rho} - v' \bar{w}' \right) \\
\frac{D \bar{w}}{Dt} &= -g \cos \theta_0 - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left( \frac{\mu \frac{\partial \bar{w}}{\partial x}}{\rho} - w' \bar{u}' \right) + \frac{\partial}{\partial y} \left( \frac{\mu \frac{\partial \bar{w}}{\partial y}}{\rho} - w' \bar{v}' \right) + \frac{\partial}{\partial z} \left( \frac{\mu \frac{\partial \bar{w}}{\partial z}}{\rho} - w' \bar{w}' \right)
\end{align*}
\]

• We make the following assumptions:
  
  • \( z = d_0 \) defines the free surface
  
  • Flow is steady: \( \frac{\partial}{\partial t} \to 0 \)
  
  • Flow is uniform in the \( x \)-direction: \( \bar{u}(z) \) and \( \frac{\partial \bar{u}}{\partial x} = 0 \).
  
  • No flow in the \( y \) direction: \( \bar{v} = 0 \)
  
  • All flow conditions are uniform in \( y \): \( \frac{\partial}{\partial y} \to 0 \)
  
  • Continuity can be used to show that the \( z \)-component of velocity is zero: \( \bar{w} = 0 \)
• With the simplifying assumptions, the Reynold’s equations reduce to

\[
0 = g \sin \theta_0 - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} (-u' u') + \frac{\partial}{\partial y} (-u' v') + \frac{\partial}{\partial z} \left( \frac{\mu \partial P}{\partial z} - u' w' \right)
\]  
(2.4.4)

\[
0 = \frac{\partial}{\partial x} (-v' u') + \frac{\partial}{\partial y} (-v' v') + \frac{\partial}{\partial z} (-v' w')
\]  
(2.4.5)

\[
0 = -g \cos \theta_0 - \frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\partial}{\partial x} (-w' u') + \frac{\partial}{\partial y} (-w' v') + \frac{\partial}{\partial z} (-w' w')
\]  
(2.4.6)

• In order to simplify the turbulent cross correlations in the Reynold’s equations, we apply the Prandtl-Boussinesq turbulence model which indicates:

\[
\frac{\tau_{ij}^t}{\rho} = -u_i^t u_j^t = \frac{\mu_t}{\rho} \left( \frac{\partial P_i}{\partial x_j} + \frac{\partial P_j}{\partial x_i} \right)
\]  
(2.4.7)

• This is a component of our constitutive model.
• This model transfers momentum through shear gradients.
• It does not transfer momentum through layers with equal turbulent time-averaged velocities.

p. 2.4.3

• The turbulent viscosity is now dependent on the intensity of turbulence. This depends on factors such as:
  • Distance from the wall
  • Flow intensity and gradients.

• Thus, looking at each cross correlation term:

\[
\overline{-u' u'} = \frac{\mu_t}{\rho} \left( 2 \frac{\partial \bar{P}}{\partial x} \right) = 0
\]  
(2.4.8)

\[
\overline{-u' v'} = \frac{\mu_t}{\rho} \left( \frac{\partial \bar{P}}{\partial y} + \frac{\partial \bar{P}}{\partial x} \right) = 0
\]  
(2.4.9)

\[
\overline{-u' w'} = \frac{\mu_t}{\rho} \left( \frac{\partial \bar{P}}{\partial z} + \frac{\partial \bar{P}}{\partial x} \right) = \frac{\mu_t \partial \bar{P}}{\rho \partial z}
\]  
(2.4.10)

\[
\overline{-v' v'} = \frac{\mu_t}{\rho} \left( 2 \frac{\partial \bar{P}}{\partial y} \right) = 0
\]  
(2.4.11)

\[
\overline{-v' w'} = \frac{\mu_t}{\rho} \left( \frac{\partial \bar{P}}{\partial z} + \frac{\partial \bar{P}}{\partial y} \right) = 0
\]  
(2.4.12)

\[
\overline{-w' w'} = \frac{\mu_t}{\rho} \left( 2 \frac{\partial \bar{P}}{\partial z} \right) = 0
\]  
(2.4.13)
• Substituting these terms into our reduced form of the Reynolds equation:

\[ 0 = g \sin \theta_0 - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial z} \left( \mu \frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho} \frac{\mu_z \partial \bar{u}}{\partial z} \right) \]  (2.4.14)

\[ 0 = 0 \]  (2.4.15)

\[ 0 = -g \cos \theta_0 - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} + \frac{\partial}{\partial x} \left( \kappa \frac{\mu \partial \bar{u}}{\partial z} + \kappa \frac{\partial \bar{u} \mu_z}{\partial z} \right) \]  (2.4.16)

• We must now complete the closure of our constitutive model by establishing a relationship for \( \mu_z \).

• There are a wide variety of models for \( \mu_z \). Some examples are:
  
  • The shear velocity - wall distance model: Turbulent viscosity is correlated to proportioned distance from the wall and bottom stress acts as a measure of turbulence intensity.

  \[ \mu_z = \rho \kappa z u_* \]  (2.4.17)

  where \( \kappa = \text{Von Karman constant} = 0.40 \) for clean water

  \[ z = \text{distance from the bottom} \]  (2.4.18)

  \[ u_* = \sqrt{\frac{\tau_0}{\rho}} = \text{shear velocity where } \tau_0 = \text{bottom stress} \]  (2.4.19)

• Prandtl Model: Turbulent viscosity is correlated to proportioned distance from the wall squared and gradient in \( \bar{u} \) velocity acts as a measure of turbulence intensity.

\[ \mu_z = \rho \kappa^2 \varepsilon \left[ \frac{\partial \bar{u}}{\partial z} \right] \]  (2.4.20)

• Selecting the first model for simplicity, our reduced Reynolds’ equations become:

\[ 0 = g \sin \theta_0 - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial z} \left( \mu \frac{\partial \bar{u}}{\partial z} + \frac{1}{\rho} \frac{\mu_z \partial \bar{u}}{\partial z} \right) \]  (2.4.21)

\[ 0 = -g \cos \theta_0 - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} + \frac{\partial}{\partial x} \left( \kappa \frac{\mu \partial \bar{u}}{\partial z} + \kappa \frac{\partial \bar{u} \mu_z}{\partial z} \right) \]  (2.4.22)

Solving for \( p \) using the Hydrostatic Pressure Equation and Simplifying the x-direction Reynolds Equation

• Noting that the turbulent viscous term in the \( z \)-direction momentum equation is a function of \( z \) only, this equation reduces to the pure hydrostatic equation:

\[ -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} = g \cos \theta_0 \]  (2.4.23)
• Integrating the hydrostatic equation between the surface $z = d_0$ where $\bar{p} = 0$ (assuming constant atmospheric pressure) and a point under the free surface at height $z$:

$$\bar{p} = 0 \quad z = d_0$$

$$\int_{\bar{p}}^{z} \partial \bar{p} = - \int_{z}^{d_0} g \cos \theta_0 dz$$  \hspace{1cm} (2.4.25)

• This leads to the pressure equation over the vertical:

$$\bar{p} = g \cos \theta_0 (d_0 - z)$$  \hspace{1cm} (2.4.26)

• Taking the $x$-derivative and noting that $d_0 = d - z$:

$$\frac{\partial \bar{p}}{\partial x} = 0$$  \hspace{1cm} (2.4.27)

• Substituting into our $x$-direction Reynold’s equation and noting that $S_0 \equiv \sin \theta_0$:

$$-gS_0 = \frac{\partial}{\partial z} \left( \frac{\mu \partial \bar{p}}{\rho \partial z} + \kappa \partial \bar{u} + \partial \bar{p} \right)$$  \hspace{1cm} (2.4.28)

Solving the $z$-direction Reynolds Equation for Stress and Reynolds Averaged Velocity

• We now assume that turbulent momentum transfer is significantly greater than molecular momentum transfer:

$$\frac{\mu \partial \bar{u}}{\rho \partial z} = \frac{\kappa_z u}{\partial z} \frac{\partial \bar{p}}{\partial z}$$  \hspace{1cm} (2.4.29)

• This is valid through-out the water column except within the viscous sublayer.

• The viscous sublayer is very small compared to $d_0$ and we will neglect it since its effect is small.

• This simplification does imply that the $\bar{p}(z)$ does not equal to zero at $z = 0$. 

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• Thus we will solve the differential equation:

\[-gS_0 = \frac{\partial}{\partial z} \left( \kappa z u_* \frac{\partial \bar{p}}{\partial z} \right)\]  \hspace{1cm} (2.4.30)

• We note that by the way we defined our constitutive relationships (including neglecting the viscous sublayer), shear stress is computed as:

\[\frac{\tau_{zx}}{\rho} \equiv \kappa z u_* \frac{\partial \bar{p}}{\partial z}\]  \hspace{1cm} (2.4.31)

• We note that at the free surface, there is no shear stress:

\[\left. \frac{\tau_{zx}}{\rho} \right|_{z = d_0} = \kappa z u_* \frac{\partial \bar{p}}{\partial z} \bigg|_{z = d_0} = 0\]  \hspace{1cm} (2.4.32)

• We can integrate our d.e. (2.4.30) between a point \( z \) and the free surface:

\[-gS_0 d_0 = \int z \left( \kappa z u_* \frac{\partial \bar{p}}{\partial z} \right) dz \]  \hspace{1cm} (2.4.33)

• Completing the integration and switching the order of the terms:

\[\kappa z u_* \frac{\partial \bar{p}}{\partial z} = gS_0(d_0 - z)\]  \hspace{1cm} (2.4.34)

• Noting that \( \tau_{zx} \) is defined in (2.4.32), stress over the water column varies as:

\[\frac{\tau_{zx}}{\rho} = gS_0(d_0 - z)\]  \hspace{1cm} (2.4.35)

• Thus stress varies linearly between zero at the free surface and having a maximum value at the bottom.
• We can re-arrange our simplified d.e. (2.4.34)

\[ \frac{\partial \bar{u}}{\partial z} = \frac{g S_0}{K u_s} \left( \frac{d_0}{z} - 1 \right) \] (2.4.36)

• Now integrate (2.4.36) between some location at the bottom \( z_b \) where \( \bar{u} = 0 \) and some point \( z \).

\[
\int_{z_b}^{z} \frac{\partial \bar{u}}{\partial z} \, dz = \frac{g S_0}{K u_s} \left( \frac{d_0}{z} - 1 \right) dz
\] (2.4.37)

\[
\bar{u} = \frac{g S_0}{K u_s} \left[ d_0 \ln z - z \right]_{z_b}^{z}
\] (2.4.38)

\[
\bar{u} = \frac{g S_0}{K u_s} \left[ d_0 \ln z - z - d_0 \ln z_b + z_b \right]
\] (2.4.39)

\[
\bar{u} = \frac{g S_0}{K u_s} \left[ d_0 \ln \left( \frac{z}{z_b} \right) - z + z_b \right]
\] (2.4.40)

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**Computing a Depth Averaged Velocity in Terms of Maximum Channel Velocity**

• Let’s now average \( \bar{u} \) over the vertical:

\[
\bar{u} = \frac{1}{d_0} \int_{z_b}^{d_0} \bar{u} \, dz
\] (2.4.41)

\[
\bar{u} = \frac{1}{d_0} \frac{g S_0}{d_0 K u_s} \int_{z_b}^{d_0} \left( d_0 \ln \left( \frac{z}{z_b} \right) - z + z_b \right) \, dz
\] (2.4.42)

\[
\bar{u} = \frac{1}{d_0} \frac{g S_0}{d_0 K u_s} \left\{ d_0 \ln z - d_0 \ln z_b - z + z_b \right\} \, dz
\] (2.4.43)

\[
\bar{u} = \frac{1}{d_0} \frac{g S_0}{d_0 K u_s} \left[ d_0 \left( \ln z - z \right) - d_0 \ln (z_b) z - \frac{z^2}{2} + z_b z \right]_{z_b}^{d_0}
\] (2.4.44)

\[
\bar{u} = \frac{1}{d_0} \frac{g S_0}{d_0 K u_s} \left[ z \left( d_0 \ln z - d_0 - d_0 \ln z_b - z + z_b \right) \right]_{z_b}^{d_0}
\] (2.4.45)

\[
\bar{u} = \frac{1}{d_0} \frac{g S_0}{d_0 K u_s} \left[ z \left( d_0 \ln \left( \frac{z}{z_b} \right) - z + z_b - d_0 + \frac{z^2}{2} \right) \right]_{z_b}^{d_0}
\] (2.4.46)
• Noting from equation (2.4.40) that

\[
\frac{\bar{u}}{\left(\frac{gS_0}{\kappa u_x}\right)} = d_0 \ln \frac{z}{z_b} - z + z_b
\]  

(2.4.47)

• Substituting

\[
\bar{u} = \frac{1}{d_0 \kappa u_x} \left[ z \frac{\bar{u}(z)}{\left(\frac{gS_0}{\kappa u_x}\right)} - zd_0 + \frac{z^2}{2} \right]_{z_b}
\]

(2.4.48)

\[
\bar{u} = \frac{1}{d_0 \kappa u_x} \left[ d_0 \frac{\bar{u}(d_0)}{\left(\frac{gS_0}{\kappa u_x}\right)} - d_0^2 + \frac{d_0^2 - z_b}{2} - z_b d_0 - \frac{z_b^2}{2} + z_b d_0 - \frac{z_b^2}{2} \right]
\]

(2.4.49)

• Note that \(\bar{u}(d_0) = \bar{u}_{\max}\) and \(\bar{u}(z_b) = 0\):

\[
\bar{u} = \frac{1}{d_0 \kappa u_x} \left[ d_0 \frac{\bar{u}_{\max}}{gS_0} - d_0 + z_b d_0 - \frac{z_b^2}{2} \right]
\]

(2.4.50)

• Further reworking the previous equation:

\[
\bar{u} = \bar{u}_{\max} + \frac{1}{d_0 \kappa u_x} \left[ d_0 - d_0^2 + z_b d_0 - \frac{z_b^2}{2} \right]
\]

(2.4.51)

• Re-arranging this equation:

\[
\bar{u} = \bar{u}_{\max} - \frac{1}{2 \kappa u_x} \left( d_0 - z_b \right)^2
\]

(2.4.52)

• Since for \(d_0 \gg z_b\)

\[
\frac{(d_0 - z_b)^2}{d_0} \approx \frac{d_0^2}{d_0} \equiv d_0
\]

(2.4.53)

• This lead us to relations between the maximum profile velocity and the depth averaged velocity:

\[
\bar{u} = \bar{u}_{\max} - \frac{1}{2} \frac{gS_0 d_0}{\kappa u_x}
\]

(2.4.54)

\[
\bar{u}_{\max} = \bar{u} + \frac{1}{2} \frac{gS_0 d_0}{\kappa u_x}
\]

(2.4.55)
Relating Channel Velocity to Channel Depth Averaged Velocity

- We note that we could have also integrated our d.e. (2.4.36) between the surface and z as

\[
\bar{u}_{\text{max}} - \bar{u} = \frac{g S_0}{K u_s} \left[ d_0 \ln z - z \right] \frac{d_0}{z} \tag{2.4.57}
\]

\[
\bar{u}_{\text{max}} - \bar{u} = \frac{g S_0}{K u_s}[d_0 \ln d_0 - d_0 - d_0 \ln z + z] \tag{2.4.58}
\]

\[
\bar{u} - \bar{u}_{\text{max}} = \frac{g S_0}{K u_s}[d_0 \ln d_0 - d_0 \ln d_0 - z + d_0] \tag{2.4.59}
\]

\[
\bar{u} = \bar{u}_{\text{max}} + \frac{g S_0}{K u_s} \left[ d_0 \ln \left( \frac{z}{d_0} \right) - z + d_0 \right] \tag{2.4.60}
\]

Now let's substitute in our relationship for \( \bar{u}_{\text{max}} \) and \( \bar{u} \) from (2.4.55) into (2.4.60):

\[
\bar{u}(z) = \bar{u} + \frac{1}{2} \frac{g S_0}{K u_s} \left[ d_0 \ln \left( \frac{z}{d_0} \right) - z + d_0 \right] \tag{2.4.61}
\]

\[
\bar{u}(z) = \bar{u} + \frac{g S_0}{K u_s} \left[ \frac{d_0}{2} + d_0 \ln \left( \frac{z}{d_0} \right) - z + d_0 \right] \tag{2.4.62}
\]

\[
\bar{u}(z) = \bar{u} + \frac{g S_0}{K u_s} \left[ \frac{3}{2} d_0 + d_0 \ln \left( \frac{z}{d_0} \right) - z \right] \tag{2.4.63}
\]

Deviations in Reynolds Averaged Velocity from the Depth Averaged Velocity Normalized by the Depth Averaged Velocity

- Now let's work out a relationship for the deviation in velocity from the mean relative to the mean:

\[
\hat{u}(z) = \bar{u}(z) - \bar{u} \tag{2.4.64}
\]

\[
\frac{\hat{u}}{\bar{u}} = \frac{\bar{u} - \bar{u}}{\bar{u}} = \frac{\bar{u}}{\bar{u}} - 1 \tag{2.4.65}
\]
• Substituting in for our relationship (2.4.63) between $\bar{u}$ and $\hat{u}$:

$$\frac{\hat{u}}{\bar{u}} = \frac{\bar{u}}{\hat{u} \kappa u_0} \left[ \frac{3}{2} d_0 + d_0 \ln \left( \frac{z}{d_0} \right) - z \right] - 1 \quad (2.4.66)$$

• This leads to an equation for the velocity deviation from the depth averaged velocity normalized by the depth averaged velocity:

$$\frac{\hat{u}}{\bar{u}} = \frac{g S_0 d_0 \left[ \frac{3}{2} + \ln \left( \frac{z}{d_0} \right) - \frac{z}{d_0} \right]}{\kappa u_0} \quad (2.4.67)$$

**Deviation in Reynolds Averaged Velocity from the Depth Averaged Velocity Normalized to the Depth Averaged Velocity Using only Channel Parameters**

• Now, we would like to relate the shear velocity $u_s$ to the bottom stress. We note from equation (2.4.35)

$$\frac{\tau_{xz}}{\rho} = g S_0 (d_0 - z) \quad (2.4.68)$$

• Evaluating this equation at the bottom:

$$\left. \frac{\tau_{xz}}{\rho} \right|_{z = z_b} = g S_0 (d_0 - z_b) \quad (2.4.69)$$

• However we note that

$$\left. \frac{\tau_{xz}}{\rho} \right|_{z = z_b} \approx \frac{\tau_0}{\rho} \quad (2.4.70)$$

• Furthermore we can readily approximate:

$$d_0 - z_b \approx d_0 \quad (2.4.71)$$

• Thus (2.4.69) simplifies to:

$$\frac{\tau_0}{\rho} = g S_0 d_0 \quad (2.4.72)$$

• Using the definition for shear velocity, this implies that:

$$u_s = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{g S_0 d_0} \quad (2.4.73)$$
• Furthermore, we previously related bottom stress to $\bar{u}$ using the constitutive relationship.

$$\tau_0 \rho = \frac{f_D W}{8} \bar{u}^2$$  \hspace{1cm} (2.4.74)

• Noting the definition for shear velocity

$$u_*^2 = \frac{f_D W}{8} \bar{u}^2$$  \hspace{1cm} (2.4.75)

• Using equation (2.4.73), we can substitute in for $u_* = \sqrt{gS_0d_0}$. Thus:

$$gS_0d_0 = \frac{f_D W}{8} u_*^2$$  \hspace{1cm} (2.4.76)

• Now we can solve for depth averaged velocity in terms of the channel slope, depth and friction factor:

$$\bar{u} = \frac{8gS_0d_0}{\kappa f_D W}$$  \hspace{1cm} (2.4.77)

• Substituting in for $\bar{u}$ and $u_*$ into our relationship for $\frac{\bar{u}}{u}$, equation (2.4.67):

$$\frac{\bar{u}}{u} = \frac{gS_0d_0}{\kappa} \left[ \frac{\sqrt{\frac{3}{2} + \ln\left(\frac{z}{d_0}\right)} - \frac{z}{d_0}}{\sqrt{\frac{3}{2} + \ln\left(\frac{z}{d_0}\right)} - \frac{z}{d_0}} \right]$$  \hspace{1cm} (2.4.78)

• This simplifies to our final relationship for the ratio $\frac{\bar{u}}{u}$ in terms of water depth and friction factor:

$$\frac{\bar{u}(z)}{u(z)} = \frac{1}{\kappa} \left[ \frac{f_D W}{8} \left( \frac{3}{2} + \ln\left(\frac{z}{d_0}\right) - \frac{z}{d_0} \right) \right]$$  \hspace{1cm} (2.4.79)
• Plotting $\frac{\tilde{u}(z)}{\bar{u}(z)}$ versus $\frac{z}{d_0}$ for $f = 0.01$, $f = 0.04$, and $f = 0.10$:
  
  • These friction values correspond to low, medium and high friction in an open channel.

• From this plot we note:
  
  • That the maximum velocities exceed the mean by 4%, 9% and 14% respectively.
  • The logarithmic profile of the velocity distributions.
  • The depth averaged velocity approximation $\bar{u}$ appears to be an excellent estimate of the actual depth varying velocity $\bar{u}(z)$ except very near the bottom.
  • The higher friction, the more that the velocity distribution deviates from the mean $\bar{u}$.
Practical Velocity Distribution Function in Wide Open Channels

• The velocity distribution for fully developed turbulent open channel flow is given approximately by Prandtl’s power law:

\[
\frac{\bar{u}(z)}{u_{\text{max}}} = \left(\frac{z}{d}\right)^{N/3}
\]  

where \( d \) = depth of flow  

\[ (2.4.80) \]

• The value of \( N \) depends on boundary friction and cross section shape.
• \( N \) ranges from 4 (for shallow water in wide rough channels) up to 12 (for smooth narrow channels).
• For uniform equilibrium flows:

\[
N = \kappa \sqrt{\frac{8}{f_D W}}
\]

where \( \kappa = 0.4 \)  

\[ (2.4.82) \]

• Most commonly used values are \( N=6 \) or \( N=7 \).

Implications of Depth Varying Velocity Profiles for the Shallow Water Equations

• Let us examine the effect of \( z \) variability on the shallow water equations.

• For the depth averaged continuity equation, the depth variability has been averaged out. Thus we can simply use the equation as is:

\[
\frac{\partial n}{\partial t} + \frac{\partial (\bar{u}H)}{\partial x} + \frac{\partial (\bar{v}H)}{\partial y} = 0
\]

\[ (2.4.84) \]

• For the depth averaged momentum equations, the dispersion terms appeared:

\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -g \frac{\partial \eta}{\partial x} + \frac{1}{H} \frac{\partial}{\partial x} \left( \int_{-h}^{\eta} \frac{\tau_{xx}}{\rho} - \bar{u}^2 \right) dz + \frac{1}{H} \frac{\partial}{\partial y} \left( \int_{-h}^{\eta} \frac{\tau_{xy}}{\rho} - \bar{u} \bar{v} \right) dz + \frac{1}{\rho} \frac{\tau_x}{H} - \frac{1}{\rho} \frac{\tau_x}{H}
\]

\[ (2.4.85) \]

\[
\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} = -g \frac{\partial \eta}{\partial y} + \frac{1}{H} \frac{\partial}{\partial x} \left( \int_{-h}^{\eta} \frac{\tau_{xy}}{\rho} - \bar{u} \bar{v} \right) dz + \frac{1}{H} \frac{\partial}{\partial y} \left( \int_{-h}^{\eta} \frac{\tau_{yy}}{\rho} - \bar{v}^2 \right) dz + \frac{1}{\rho} \frac{\tau_y}{H} - \frac{1}{\rho} \frac{\tau_y}{H}
\]

\[ (2.4.86) \]
• We note that for steady uni-directional steady uniform open channel flow
\[ \bar{u} = \bar{u}(z) \text{ only} \] \[ (2.4.87) \]
\[ \bar{v} = 0 \] \[ (2.4.88) \]

• Thus it is clear that for uniform flow the dispersion portions of the lateral/dispersion terms drop out.

\[
\frac{1}{H} \frac{\partial}{\partial x} \int_{-h}^{\eta} (-\bar{u}^2) dz = 0 \] \[ (2.4.89) \]
\[
\frac{1}{H} \frac{\partial}{\partial y} \int_{-h}^{\eta} (-\bar{u}\bar{v}) dz = 0 \] \[ (2.4.90) \]
\[
\frac{1}{H} \frac{\partial}{\partial x} \int_{-h}^{\eta} (-\bar{v}^2) dz = 0 \] \[ (2.4.91) \]
\[
\frac{1}{H} \frac{\partial}{\partial y} \int_{-h}^{\eta} (-\bar{v}^2) dz = 0 \] \[ (2.4.92) \]

• In fact we can also demonstrate that the lateral diffusion terms drop out assuming a Prandtl-Boussinesq-type closure (e.g., see our derivation of the vertical velocity profile in a channel).

• It makes sense that the lateral diffusion/dispersion terms drop out for a uniform flow in the x-direction (despite \( \bar{u} = \bar{u}(z) \)) since all these terms originate from the advective terms. Since the advective terms are zero in a uniform flow, the associated diffusion/dispersion terms will be zero.

• For non-uniform flows, the advective terms and thus the lateral diffusion/dispersion terms will become a portion of the momentum balance.
  
  • Their importance is definitely correlated to the importance of the advective terms.
  
  • Due to the logarithmic nature of the depth-varying component of the velocity profile, \( \bar{u} \), the dispersion terms are often not significant.
  
  • If these terms are significant, they can be explicitly computed by either assuming or calculating a vertical velocity profile. This is what is done in sophisticated, state-of-the-art shallow water equation codes.
Implication of Depth-Varying Velocity Profiles for the Bernoulli Equation

- We must adjust the kinetic energy term in the total energy when using depth averaged or cross sectional mean flow as the variable.

\[ H_E = \frac{P}{g} + z + \frac{u^2}{2g} \]  
(2.4.93)

- The true mean velocity head is found by adjusting this term by:

\[ \alpha = \frac{\int \bar{u}^3 dA}{\bar{u}^3 A} \]  
(2.4.94)

- For the \( \frac{1}{N} \) power law velocity distribution, the kinetic energy coefficient equals

\[ \alpha = \frac{(N + 1)^3}{N^3 (N + 3)} \]  
(2.4.95)

- We note that for a typical value of \( N = 7 \),

\[ \alpha = 1.045 \]  
(2.4.96)

- Note that this concept can be extended to channels with widely varying flow properties and that the averaging can be accomplished on a sectional basis:

\[ \bar{u}_m = \frac{\bar{u}_1 A_1 + \bar{u}_2 A_2 + \bar{u}_3 A_3}{A_1 + A_2 + A_3} \]  
(2.4.97)

\[ \alpha = \frac{\bar{u}_1^3 A_1 + \bar{u}_2^3 A_2 + \bar{u}_3^3 A_3}{\bar{u}_m^3 (A_1 + A_2 + A_3)} \]  
(2.4.98)
• For all cases, when using depth averaged or cross sectionally averaged flow, the total energy becomes:

\[ H_E = \frac{\rho}{\gamma} z + \alpha \frac{\bar{u}^2}{2g} \]  

(2.4.99)

• We note that for uniform flow, \( \bar{u}_1 = \bar{u}_2 \), and therefore the kinetic energy terms do not come into play.
  • This makes sense since the advective acceleration terms are zero and the advective terms are the origin of the kinetic energy terms.

• Thus for uniform flow, the correction factor \( \alpha \) becomes an inconsequential point.

• For non-uniform flow, we must compute or assume a velocity profile and estimate a value of \( \alpha \).

• Often the variation in \( \alpha \) is small compared to uncertainties such as frictional resistance.

Implication of Depth-Varying Velocity Profiles for Momentum Conservation for Finite Control Volumes

• We must adjust the momentum flux term in the conservation of momentum applied to a finite control volume.

\[ \frac{\partial}{\partial t} \iint_{CV} \rho U dV + \iint_{CS} U(\rho \cdot \hat{n}) dA = \iint_{CV} T dA + \iint_{CV} B \rho dV \]  

(2.4.100)

• For a channel, the momentum flux term simplified to

\[ \iint_{CS} \bar{u}(\rho \bar{u}) dA \]  

(2.4.101)

• We can use depth averaged velocity \( \bar{u} \) in this expression by computing an adjustment coefficient:

\[ \beta = \frac{\Lambda}{\rho \bar{u}^2 A} \]  

(2.4.102)
• For the \( \frac{1}{N} \) power law velocity distribution, the momentum correction coefficient equals

\[
\beta = \frac{(N + 1)^2}{N(N + 2)} \tag{2.4.103}
\]

• We note that for a typical value of \( N = 7 \)

\[
\beta = 1.016 \tag{2.4.104}
\]

• The momentum flux term for an open channel can be computed as

\[
\int \int_{c,s} \bar{n}(\rho \bar{u}) dA = \rho Q(\beta_2 \bar{u}_2 - \beta_1 \bar{u}_1) \tag{2.4.105}
\]

• Again we note that for uniform flow, \( \bar{u}_1 = \bar{u}_2 \) and therefore the momentum flux terms do not come into play.
  
  • Again, this makes sense since the momentum flux terms are related to the advective acceleration terms (they advect momentum). Since the advective terms for a uniform flow are zero, the momentum flux terms in the surface integral must be zero.

• Thus, for uniform flow, the momentum flux correction factor, \( \beta \), does not come into play.

• For non-uniform flow, we must compute or assume a velocity profile and estimate a value of \( \beta \).

• Often the variation in \( \beta \) is small compared to uncertainties such as frictional resistance.