UNIVERSITY OF NOTRE DAME
Department of Civil and Environmental Engineering and Earth Sciences

CE 60130 Finite Elements in Engineering — **January 25th, 2018**
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Homework Set #1

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$10\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5$$
$$u|_{x=10} = 15$$

Solve this ODE with the method of weighted residuals using polynomial approximation basis functions. Recall that the approximate solution follows

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

where u_B is the boundary function that satisfies the boundary conditions, α_i are the unknown coefficients which must be determined and $\phi_i(x)$ are the known basis functions that form a complete set.

a) *Formulate the boundary function u_B such that the boundary conditions are satisfied*

b) *Show that the basis functions*

$$\phi_i(x) = x^i(x - 10)$$

satisfy the admissibility and completeness criteria.

c) *Develop the weighted residual formulation for a generic operator $\mathcal{L}(u)$, a generic function $p(x)$, a generic boundary condition u_B, a generic approximation function $\phi_i(x)$, and generic weighting functions w_j.*

d) *For the case $N = 5$, organize the weighted residual formulation into a system of simultaneous equations using the Galerkin weighted residual method. Express your result as both 5 separate equations and as a matrix system $[A]_{5x5}\{x\}_{5x1} = \{b\}_{5x1}$. Keep using the generic variables (do NOT make the substitutions $\phi_i(x) = x^i(x - 10)$, $\mathcal{L} = 10\frac{d^2}{dx^2} + \frac{d}{dx} + \frac{1}{6}$, $p(x) = -3$,* in this step)

e) *For the given differential equation, $\phi_i(x)$, your selection of u_B, implement the Galerkin residual method, where $w_j(x) = \phi_j(x)$. Analytically work out the j^{th} row and the i^{th} column entry of the matrix and the i^{th} column entry of the right hand side vector of your system of equations.*

f) *For the cases* $N = 1$, $N = 2$, $N = 3$, $N = 4$, *and* $N = 5$, *fill in the numerical values of the matrix and the right hand side load vector and find the corresponding* α *vector for each case.*

g) *For the cases* $N = 1$, $N = 2$, $N = 3$, $N = 4$, *and* $N = 5$, *find* u_{app} *and plot the approximated solution in the same figure. Also include in the figure the analytical solution of the ODE. Comment on the results as you increase* N. *The analytical solution of the ODE is:*

$$u(x) = -e^{-x/20}\left(18e^{x/20} - 23\cos\left(\frac{1}{20}\sqrt{\frac{17}{3}}x\right) + 23\cot\left(\frac{\sqrt{\frac{17}{3}}}{2}\right)\sin\left(\frac{1}{20}\sqrt{\frac{17}{3}}x\right) - 33\sqrt{e}\csc\left(\frac{\sqrt{\frac{17}{3}}}{2}\right)\sin\left(\frac{1}{20}\sqrt{\frac{17}{3}}x\right)\right)$$

h) *The Condition Number* C *is the ratio of the largest to smallest singular value in the singular value decomposition of a matrix. When this value is too large (*$\log(C)$ *is greater than the precision of matrix entries), the matrix is ill-conditioned.*
For $N = 5$, $N = 10$, $N = 25$, *and* $N = 50$, *plot the approximated solution in the same figure. Compare those solutions with the analytical solution of the ODE. Use the command cond(A) in Matlab to compute the condition number* C *of the matrix* A, *and comment how* C *changes as* N *increases. How do these values affect the solution?*