UNIVERSITY OF NOTRE DAME
Department of Civil and Environmental Engineering and Earth Sciences

CE 60130 Finite Elements in Engineering **February 1st, 2018**
J.J. Westerink **Due: February 8th, 2018**

Homework Set #2 KEY

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$\frac{1}{10}\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5 \qquad \text{and} \qquad \frac{du}{dx}_{|x=10} = \frac{-3\pi}{5}$$

Its analytical solution is given by:

$$u(x) = -18e^{-10x}\left(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x}\right)$$

We will examine components of solving this ODE with the method of weighted residuals using different kind of basis functions. Recall that the approximate solution follows

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

where u_B is the boundary function that satisfies the boundary conditions, α_i are the unknown coefficients which must be determined and $\phi_i(x)$ are the known basis functions that form a complete set.

QUESTION 1

a) *Formulate the boundary function u_B such that the boundary conditions are satisfied. Hint: The boundary function must satisfy both derivative and functional value boundary conditions.*

Solution:

To begin, we find the appropriate boundary function u_B using the polynomial expansion x^n. We know we have to satisfy $u_B(x=0) = 5$ and $\left.\frac{du_B}{dx}\right|_{x=10} = \frac{-3\pi}{5}$. Consider a boundary function of the form $u_B = a_1 + a_2 x$.
We have:

$$\begin{aligned} u_B|_{x=0} &= a_1 + a_2 \times 0 \\ 5 &= a_1 + a_2 \times 0 \\ a_1 &= 5 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \left.\frac{du_B}{dx}\right|_{x=10} &= a_2 \\ a_2 &= \frac{-3\pi}{5} \end{aligned}$$

Then, u_B looks as :

$$u_B = 5 + \frac{-3\pi}{5}x$$

b) *Let us examine using a polynomial expansion for this PDE. Show that the basis functions*

$$\phi_i(x) = x^i \left(\frac{10(i+1)}{i} - x \right)$$

satisfy the admissibility and completeness criteria.

Solution:

For admissibility we need $u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i$ to satisfy the boundary conditions and the correct degree of functional continuity. So first we need ϕ_i to satisfy the homogeneous form of the boundary conditions, that is, for all i:

$$\phi_i|_{x=0} = 0 \quad \text{and} \quad \phi_i|_{x=10} = 0$$

Let: $\phi_i = x^i \left(\frac{10(i+1)}{i} - x \right)$. Then:

$$\phi_i(x=0) = 0 \quad \text{and} \quad \phi_i(x=10) = 0 \quad \forall i \in \mathcal{N}$$

Since our ODE is second order, for functional continuity we need

$$u_{app} \in W^{(2)} = C_1.$$

Since $\phi_i(x) \in W^{(\infty)}$, u_{app} satisfies the correct degree of functional continuity. For completeness, we need the ϕ_i's to form a linearly independent complete set, which means that as we add more and more terms, $u_{app} \to u$. This set of polynomial functions satisfies this requirement.

c) *For the given differential equation, $\phi_i(x)$, your selection of u_B, implement the Galerkin residual method, where $w_j(x) = \phi_j(x)$. Express the generic term of the matrix $\langle \mathcal{L}(\phi_i(x)), \phi_j(x) \rangle_\Omega$ as a function of i and j and the generic term of the vector at the right hand side as a function of j.*

Solution:

Galerkin method implies that $\phi_j = w_j$. Thus, we can substitute the w_j as follows:

$$\sum_{i=1}^{N} \alpha_i \langle \mathcal{L}(\phi_i(x)), \phi_j(x) \rangle_\Omega = \langle -\mathcal{L}(u_B) + p(x), \phi_j(x) \rangle_\Omega, \qquad j = 1, ..., N$$

The generic terms are:

$$\langle \mathcal{L}(\phi_i(x)), \phi_j(x) \rangle_\Omega \, \alpha_i = \langle -\mathcal{L}(u_B) + p(x), \phi_j(x) \rangle_\Omega \quad \forall i, j = 1, ..., N$$

where:

$$\mathcal{L}(*) = \frac{1}{10}\frac{d^2(*)}{dx^2} + \frac{d(*)}{dx} + \frac{(*)}{6} = -3, \quad u_B = 5 + \frac{-3\pi}{5}x, \quad \phi_i(x) = x^i \left(\frac{10(i+1)}{i} - x \right)$$

Plugging equations for u_B and ϕ in the differential operator and computing the generic terms of the matrix and the right hand side of the equation, we get expressions to each term, as a function of i and j:

$$\begin{aligned}
\langle \mathcal{L}(\phi_i(x)), \phi_j(x) \rangle_\Omega = &\frac{2.84 \times 10^{-14} i^4 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} + \frac{100 i^4 10^{i+j}}{j(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{498 i^3 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{2 i^3 j 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{833.333 i^3 10^{i+j}}{j(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{4 i^2 j^2 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{3354.67 i^2 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} + \frac{684 i^2 j 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{1500 i^2 10^{i+j}}{j(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{5.68 \times 10^{-14} j^4 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&- \frac{2 i j^3 10^{i+j}}{(i+j-1.)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{2 j^3 10^{i+j}}{(i+j-1.)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{333.333 j^3 10^{i+j}}{i(i+j-1(i+j)(i+j+1)(i+j+2)(i+j+3)} + \frac{284 i j^2 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{1954.67 j^2 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} + \frac{1000 j^2 10^{i+j}}{i(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&+ \frac{3344.67 i 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} + \frac{4130.67 i j 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} \\
&- \frac{833.333 i 10^{i+j}}{j(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{1178.6710^{i+j}}{(i+j-1)(i+j)(i+j+1.)(i+j+2)(i+j+3)} \\
&+ \frac{3444.67 j 10^{i+j}}{(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{100010^{i+j}}{i(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3.)} \\
&- \frac{333.333 j 10^{i+j}}{i(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)} - \frac{160010^{i+j}}{j(i+j-1)(i+j)(i+j+1)(i+j+2)(i+j+3)}
\end{aligned}$$

$$\langle -\mathcal{L}(u_B) + p(x), \phi_j(x) \rangle_\Omega = \frac{10^j \left(8.882 \times 10^{-14} j^2 + 238.643 j - 226.549 \right)}{j(j+2)(j+3)}$$

d) *Develop a code to solve each system using the Galerkin weight method. Try different values of N and compute the condition number C of the matrix. Try different values of N starting with $N = 1$ increasing it until you cannot get a solution. Create a plot N v/s $log(cond())$ and plot the different u_{app} solution for each expansion set and the analytical solution in the same figure. Comment on the results as you increase N. Does any of your solutions fit the analytical solution? If not, explain why you think it doesn't converge.*

Solution:

Figure 1 shows a comparison between the analytical solution for this problem, and the numerical solution using different number of N. Notice the solution never converges to the analytical solution due to the matrix is ill-conditioned. Figure 2 shows the variation of $log(cond())$ as N increases, and since the relation is almost linear, the values of $log(cond())$ always increases as N does.

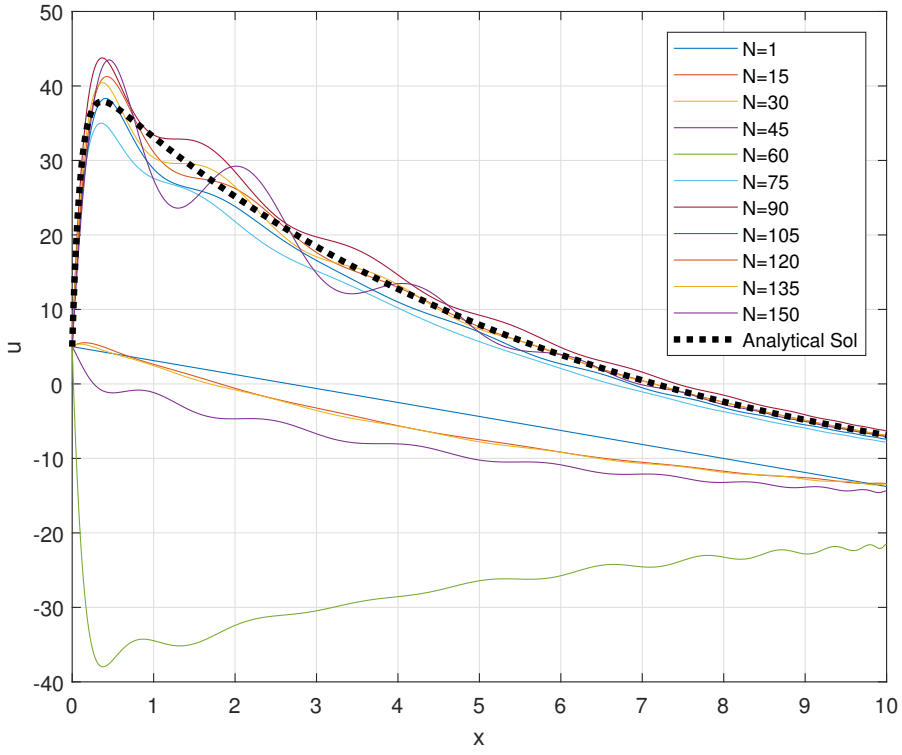

FIGURE 1. Polynomial expansions for different values of N

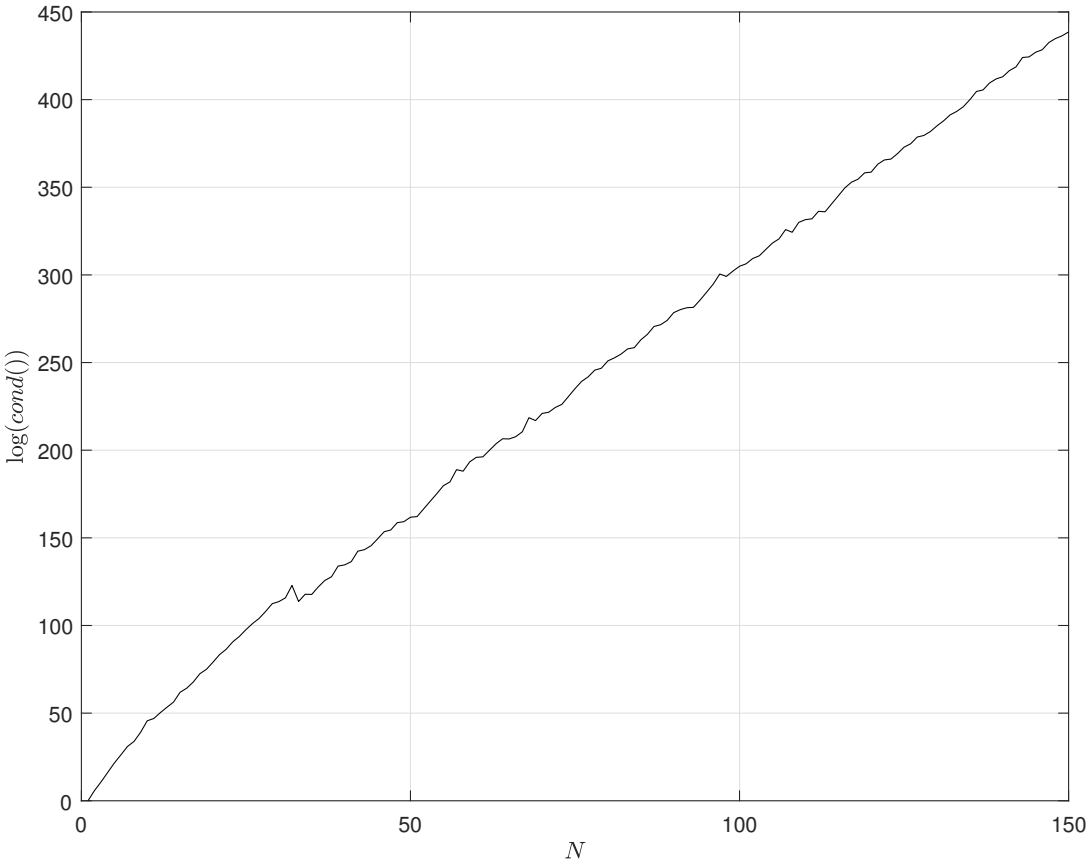

FIGURE 2. $\log C$ v/s N

e) *Determine the L_2 error by comparing with the point based analytical solution. Plot N versus $log(\mathcal{E})$. The L_2 error can be calculated as the Root Mean Square Error, as follows:*

$$\mathcal{E} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - f_i)^2}$$

where N is the number of points in the point-wise comparison, f_i is the exact value at the i^{th} point and y_i is the value given by your model at the i^{th} point.

Solution:

Figure 3 shows the evolution of the error as we increase N. As It can be noticed, the error decreases as N increases, but since the solution never converges, it never becomes a constant value closed to zero, which is mainly produced by the bad features of the matrix.

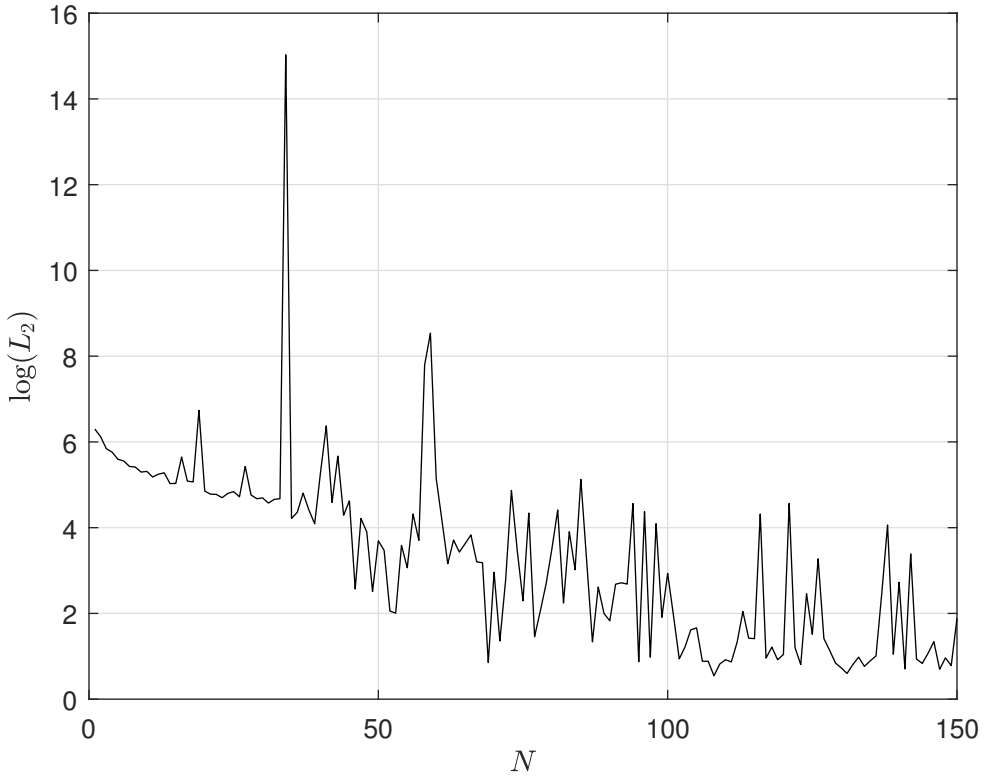

FIGURE 3. $\log L_2$ v/s N

QUESTION 2

Considering the ODE in the previous Question, now let us examine using sine/cosine expansion for this PDE. Repeat ***Question 1 b), c), d), and e)***

$$\phi_i(x) = \sin\left(\frac{1}{20}\pi(2i-1)x\right)$$

Solution:

Again, for admissibility we need $u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i$ to satisfy the boundary conditions and the correct degree of functional continuity. So first we need ϕ_i to satisfy the homogeneous form of the boundary conditions, that is, for all i:

$$\phi_i|_{x=0} = 0 \quad \text{and} \quad \phi_i|_{x=10} = 0$$

Let: $\phi_i = \sin\left(\frac{1}{20}\pi(2i-1)x\right)$. Then:

$$\phi_i(x=0) = \sin(0) = 0 \quad \text{and} \quad \phi_i(x=10) = \sin\left(\frac{\pi}{2}(2i-1)\right) = 0 \quad \forall i \in \mathcal{N}$$

For completeness, we need the ϕ_i's to form a linearly independent complete set, which means that as we add more and more terms, $u_{app} \to u$. This set of sine functions satisfies this requirement.

Applying the Galerkin method, we got generic expressions for matrix A and vector in the right hand side of the equation. In this case, we must difference the generic term when i is equal to j, or they are different. For i different to j:

$$\begin{aligned}
&\langle \mathcal{L}(\phi_i(x)), \phi_j(x) \rangle_\Omega = \\
&\quad i(0.5\ -j) + (j-0.5)\cos(1.5708 - 3.14159j)(((\frac{0.0314159i}{(i-j)(i+j-1)} - 0.0314159)i \\
&\quad -0.522662)\sin(1.5708 - 3.14159i) + (i-0.5)\cos(1.5708\ -3.14159i)) - 0.0314159(i-0.5) \\
&\quad \sin(1.5708 - 3.14159j)((15.9155 - 31.831i)\sin(1.5708 - 3.14159i) + (i(i-1) - 16.6369) \\
&\quad \cos(1.5708 - 3.14159i)) + 0.5j - 0.25
\end{aligned}$$

And, for i equal to j:

$$\begin{aligned}
&\langle \mathcal{L}(\phi_i(x)), \phi_i(x) \rangle_\Omega = \\
&\quad \frac{1}{1.i - 0.5}\,[\ i(i(0.074022\ -0.049348i) + 1.04632) + (i(0.00785398\ -0.00785398i) + 0.130666) \\
&\quad \sin(3.14159\ -6.28319i) + (0.125\ -0.25i)\cos(3.14159\ -6.28319i) - 0.535498\ ]
\end{aligned}$$

$$\langle -\mathcal{L}(u_B) + p(x), \phi_j(x) \rangle_\Omega = \frac{(1.89906\ -3.79812j)\sin(\pi j) - 6.20188j - 3.1831\cos(\pi j) + 3.10094}{(0.5\ -1.j)^2}$$

In Figure 4, we can see how the solution improves as we increase N.

In figure 5, we can see how the condition number increases as we increase N, but it seems to reach to an equilibrium for large number of modes. The log of the condition number for the first 150 modes is less than 14, which is a value that the computer can handle, so we should not expect ill-condition behavior of the matrix.
In Figure 6 the error decreases as we increase N, which makes sense because as we saw in Figure 4, the solution gets better for large numbers of N.

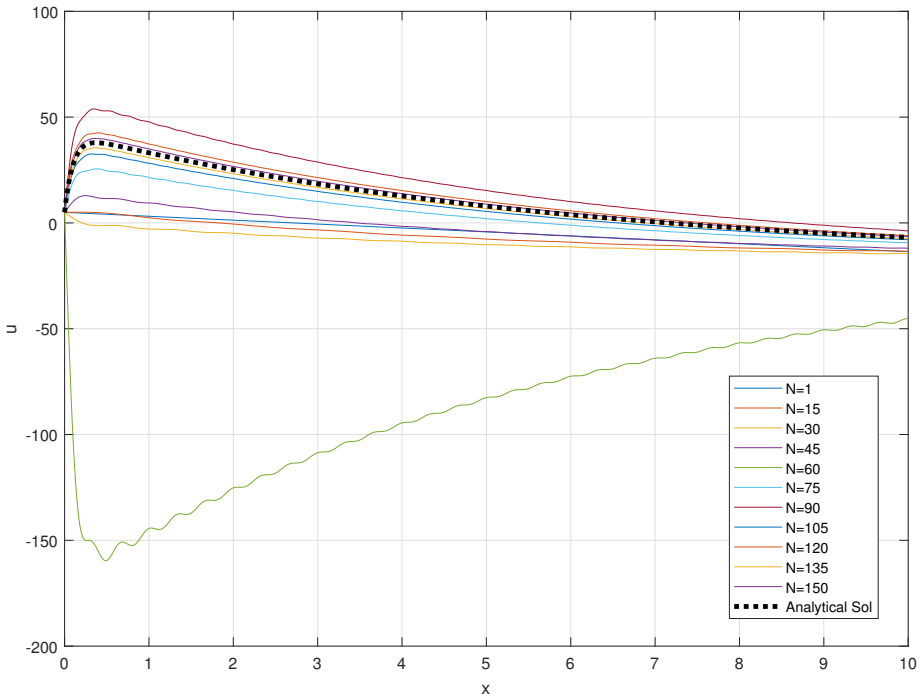

FIGURE 4. sin expansions for different values of N

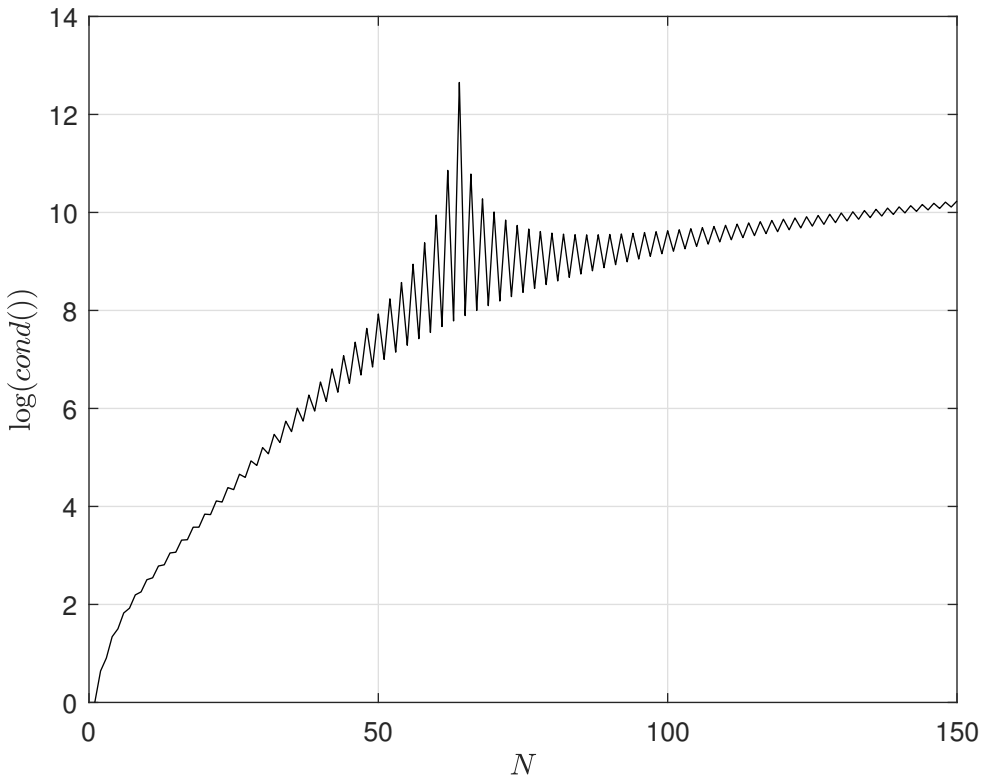

FIGURE 5. $\log C$ v/s N

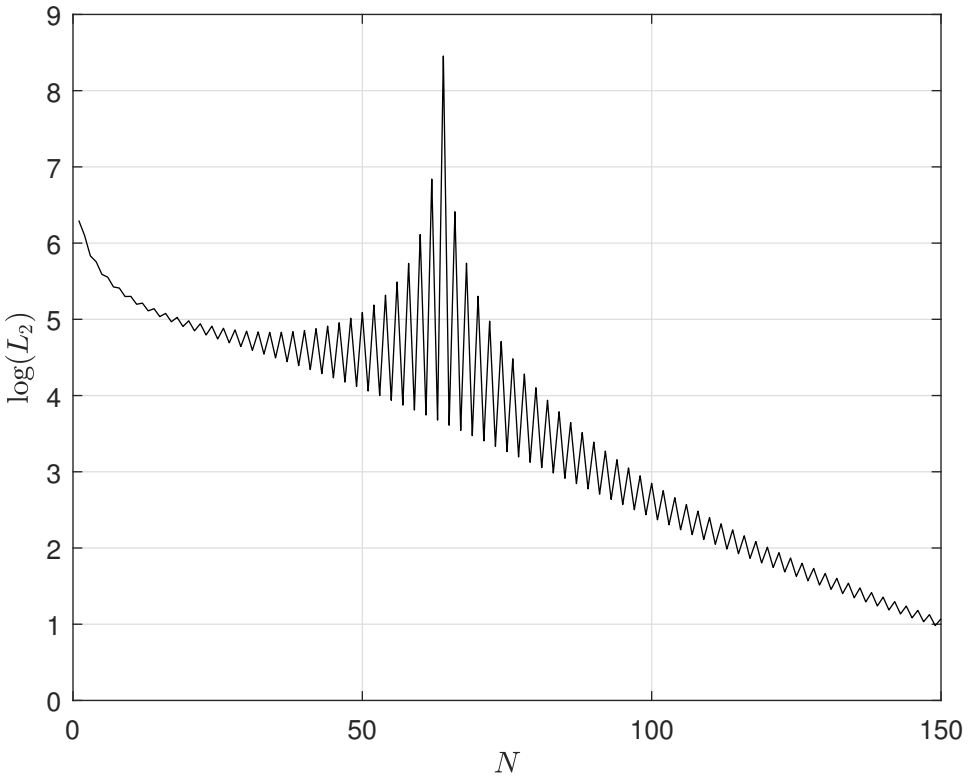

FIGURE 6. $\log L_2$ v/s N

QUESTION 3

Comment on your solution of Question 1 and 2. Which set of basis function works better?. Explain why one set of basis functions works better than the other. Does it have anything to do with the condition number of each matrix?

Solution:

If we compare solutions in question 1 (using polynomial basis functions), and question 2 (with a sine expansion function), we can notice the second approach seems to work better. The main reason of bad results we got in previous questions is based on the ill-conditioned matrix. But the bounded feature of the sine expansion helps to improve the quality of the matrix, being useful to improve the convergence and accuracy of numerical results.