UNIVERSITY OF NOTRE DAME

Department of Civil and Environmental Engineering and Earth Sciences

CE 60130 Finite Elements in Engineering — **February 25th, 2018**

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Homework Set #4

QUESTION 1

Now consider a four-node problem with specified function values

$x_1 = 0$	$f_1 = 3$
$x_2 = 1$	$f_2 = 5$
$x_3 = 2$	$f_3 = 6$
$x_4 = 4$	$f_4 = 12$

a) Derive the Lagrange interpolation function using the power series approach $g(x) = a + bx + cx^2 + dx^3$ by enforcing the constraints and solving the system of equations. Demonstrate that your solution for $g(x)$ works by evaluating $g(x_1)$, $g(x_2)$, $g(x_3)$, and $g(x_4)$.

Solution:

Specify $g(x) = a + bx + cx^2 + dx^3$. By enforcing the functional constraints, we establish a system of equations:

$$g(0) = a = 3$$
$$g(1) = a + b + c + d = 5$$
$$g(2) = a + 2b + 4c + 8d = 6$$
$$g(4) = a + 4b + 16c + 64d = 12$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 12 \end{bmatrix}$$

Solving the system of equations, we get

$$\begin{bmatrix} 3 \\ 5 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{37}{12} \\ -\frac{11}{8} \\ \frac{7}{24} \end{bmatrix}$$

Therefore, the Lagrange interpolation function is $g(x) = 3 + \frac{37}{12}x - \frac{11}{8}x^2 + \frac{7}{24}x^3$.
Verifying the result, we evaluate $g(x_1) = g(0) = 3$, $g(x_2) = g(1) = 5$, $g(x_3) = g(2) = 6$, $g(x_4) = g(4) = 12$.

b) Derive the Lagrange interpolation function using the basis functions $g(x) = f_1\phi_1(x) + f_2\phi_2(x) + f_3\phi_3(x) + f_4\phi_4(x)$. Use your knowledge, $\phi_i(x_j) = \delta_{ij}$ and the fact that for each function you know the roots and that $\phi_i(x_i) = 1$. Write and draw each $\phi_i(x)$. Demonstrate that your solution for $g(x)$ works by evaluating $g(x_1)$, $g(x_2)$, $g(x_3)$, and $g(x_4)$.

Solution:

The basis functions are:

$$\phi_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \frac{1}{8}(4-x)(-2+x)(-1+x) = -\frac{x^3}{8} + \frac{7x^2}{8} - \frac{7x}{4} + 1$$

$$\phi_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \frac{1}{4}(4-x)(-1+x)x = \frac{x^3}{3} - 2x^2 + \frac{8x}{3}$$

$$\phi_3(x) = \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} = \frac{1}{4}(4-x)(x-1)x = -\frac{x^3}{4} + \frac{5x^2}{4} - x$$

$$\phi_4(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \frac{1}{24}(x-2)(x-1)x = \frac{x^3}{24} - \frac{x^2}{8} + \frac{x}{12}$$

Draw each $phi_i(x)$:

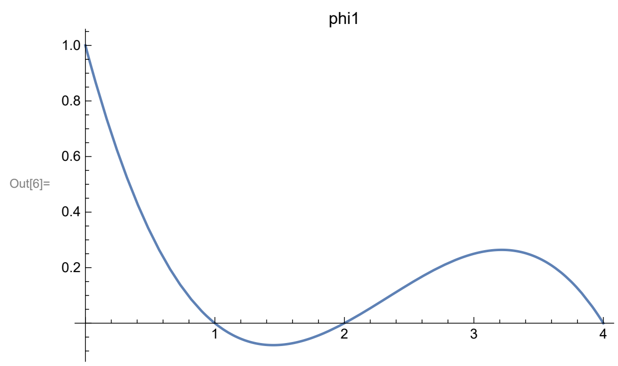

FIGURE 1. $\phi_1(x)$

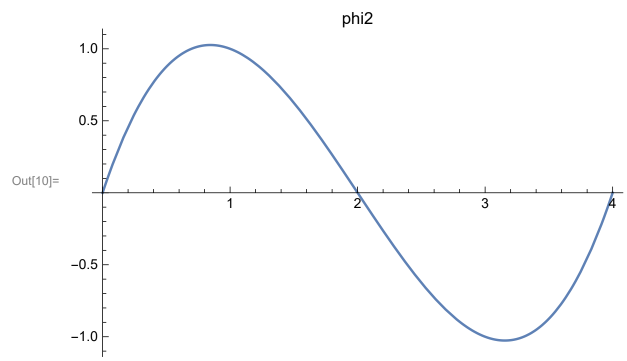

FIGURE 2. $\phi_2(x)$

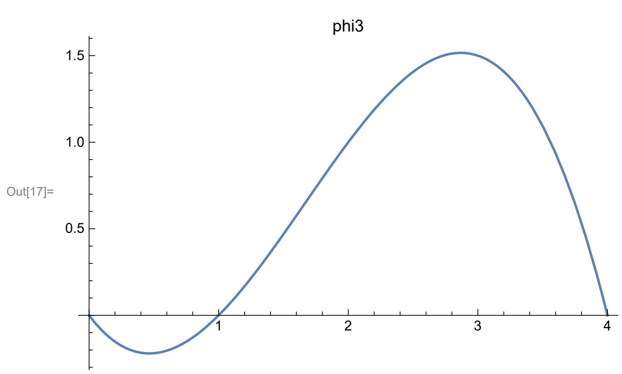

FIGURE 3. $\phi_3(x)$

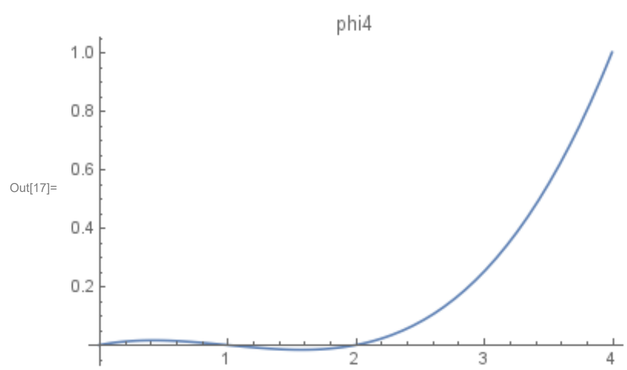

FIGURE 4. $\phi_4(x)$

Then,

$$\begin{aligned} g(x) &= f_1\phi_1(x) + f_2\phi_2(x) + f_3\phi_3(x) + f_4\phi_4(x) \\ &= 3\phi_1(x) + 5\phi_2(x) + 6\phi_3(x) + 12\phi_4(x) \end{aligned}$$

At $x = x_1$, it follows from $\phi_1(x_1) = 1$, $\phi_2(x_1) = 0$, $\phi_3(x_1) = 0$, $\phi_4(x_1) = 0$ that $g(x_1) = 3 \times 1 + 5 \times 0 + 6 \times 0 + 12 \times 0 = 3$. Similarly, we can verify $g(x_2) = 5$, $g(x_3) = 6$, $g(x_4) = 12$.

c) Simplifying your result in part b), show that it is exactly the same form as you obtained in part a).

Solution:

From result in part b),

$$\begin{aligned}
g(x) &= f_1\phi_1(x) + f_2\phi_2(x) + f_3\phi_3(x) + f_4\phi_4(x) \\
&= 3\phi_1(x) + 5\phi_2(x) + 6\phi_3(x) + 12\phi_4(x) \\
&= 3(-\frac{x^3}{8} + \frac{7x^2}{8} - \frac{7x}{4} + 1) + 5(\frac{x^3}{3} - 2x^2 + \frac{8x}{3}) \\
&\quad + 6(-\frac{x^3}{4} + \frac{5x^2}{4} - x) + 12(\frac{x^3}{24} - \frac{x^2}{8} + \frac{x}{12}) \\
&= (-\frac{3}{8} + \frac{5}{3} - \frac{3}{2} + \frac{1}{2})x^3 + (\frac{21}{8} - 10 + \frac{15}{2} - \frac{3}{2})x^2 \\
&\quad + (-\frac{21}{4} + \frac{40}{3} - 6 + 1)x + 3 \\
&= \frac{7x^3}{24} - \frac{11x^2}{8} + \frac{37x}{12} + 3
\end{aligned}$$

It is exactly the same result obtained from part a).

QUESTION 2
Consider a two point problem

$x_1 = 0$	f_1	$f_1^{(1)}$	$f_1^{(2)}$
$x_2 = 1$	f_2	$f_2^{(1)}$	$f_2^{(2)}$

a) Derive g(x) for this 2-node C_2 Hermite. Show that your g(x) matches the constraints.

Solution:
If we assume the approximating function takes the form

$$g(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$$

And

$$g^{(1)}(x) = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$$

$$g^{(2)}(x) = 2c + 6dx + 12ex^2 + 20fx^3$$

Then we get the system of equations:

$$\begin{aligned} &a = f_1 \\ &b = f_1^{(1)} \\ &2c = f_1^{(2)} \\ &a + b + c + d + e + f = f_2 \\ &b + 2c + 3d + 4e + 5f = f_2^{(1)} \\ &2c + 6d + 12e + 20f = f_2^{(2)} \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} f_1 \\ f_1^{(1)} \\ f_1^{(2)} \\ f_2 \\ f_2^{(1)} \\ f_2^{(2)} \end{pmatrix}.$$

The solution is:

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} f_1 \\ f_1^{(1)} \\ \frac{1}{2} f_1^{(2)} \\ -10f_1 + 10f_2 - 6f_1^{(1)} - 4f_2^{(1)} - \frac{3}{2} f_1^{(2)} + \frac{1}{2} f_2^{(2)} \\ 15f_1 - 15f_2 + 8f_1^{(1)} + 7f_2^{(1)} + \frac{3}{2} f_1^{(2)} - f_2^{(2)} \\ -6f_1 + 6f_2 - 3f_1^{(1)} - 3f_2^{(1)} - \frac{1}{2} f_1^{(2)} + \frac{1}{2} f_2^{(2)} \end{pmatrix}$$

The interpolation function is:

$$
\begin{aligned}
g(x) = {} & f_1 + f_1^{(1)} x + \frac{1}{2} f_1^{(2)} x^2 \\
& + (-10 f_1 + 10 f_2 - 6 f_1^{(1)} - 4 f_2^{(1)} - \frac{3}{2} f_1^{(2)} + \frac{1}{2} f_2^{(2)}) x^3 \\
& + (15 f_1 - 15 f_2 + 8 f_1^{(1)} + 7 f_2^{(1)} + \frac{3}{2} f_1^{(2)} - f_2^{(2)}) x^4 \\
& + (-6 f_1 + 6 f_2 - 3 f_1^{(1)} - 3 f_2^{(1)} - \frac{1}{2} f_1^{(2)} + \frac{1}{2} f_2^{(2)}) x^5
\end{aligned}
$$

It is straightforward to verify $g(x)$ matches all the constraints.

b) Derive and plot the basis function from part a) by factorizing all the specified data ($f_1, f_1^{(1)}, f_1^{(2)}$, $f_2, f_2^{(1)}, f_2^{(2)}$). Show that

$$\phi_i(x_j) = \delta_{ij} \qquad \phi_i^{(1)}(x_j) = 0 \qquad \phi_i^{(2)}(x_j) = 0$$
$$\psi_i(x_j) = 0 \qquad \psi_i^{(1)}(x_j) = \delta_{ij} \qquad \psi_i^{(2)}(x_j) = 0$$
$$\Theta_i(x_j) = 0 \qquad \Theta_i^{(1)}(x_j) = 0 \qquad \Theta_i^{(2)}(x_j) = \delta_{ij}$$

Solution:
After factorizing ($f_1, f_1^{(1)}, f_1^{(2)}, f_2, f_2^{(1)}, f_2^{(2)}$), we get:

$$\begin{aligned}
g(x) &= f_1 + f_1^{(1)}x + \frac{1}{2}f_1^{(2)}x^2 \\
&+ (-10f_1 + 10f_2 - 6f_1^{(1)} - 4f_2^{(1)} - \frac{3}{2}f_1^{(2)} + \frac{1}{2}f_2^{(2)})x^3 \\
&+ (15f_1 - 15f_2 + 8f_1^{(1)} + 7f_2^{(1)} + \frac{3}{2}f_1^{(2)} - f_2^{(2)})x^4 \\
&+ (-6f_1 + 6f_2 - 3f_1^{(1)} - 3f_2^{(1)} - \frac{1}{2}f_1^{(2)} + \frac{1}{2}f_2^{(2)})x^5 \\
&= f_1 \cdot (1 - 10x^3 + 15x^4 - 6x^5) + f_2 \cdot (10x^3 - 15x^4 + 6x^5) \\
&+ f_1^{(1)} \cdot (x - 6x^3 + 8x^4 - 3x^5) + f_2^{(1)} \cdot (-4x^3 + 7x^4 - 3x^5) \\
&+ f_1^{(2)} \cdot (\frac{1}{2}x^2 - \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{1}{2}x^5) + f_2^{(2)} \cdot (\frac{1}{2}x^3 - x^4 + \frac{1}{2}x^5)
\end{aligned}$$

Therefore,

$$\phi_1(x) = 1 - 10x^3 + 15x^4 - 6x^5 \qquad \phi_2(x) = 10x^3 - 15x^4 + 6x^5$$
$$\psi_1(x) = x - 6x^3 + 8x^4 - 3x^5 \qquad \psi_2(x) = \frac{1}{2}x^2 - 4x^3 + 7x^4 - 3x^5$$
$$\Theta_1(x) = \frac{1}{2}x^2 - \frac{3}{2}x^3 + \frac{3}{2}x^4 - \frac{1}{2}x^5 \qquad \Theta_2(x) = \frac{1}{2}x^3 - x^4 + \frac{1}{2}x^5$$

We plot all these basis functions:

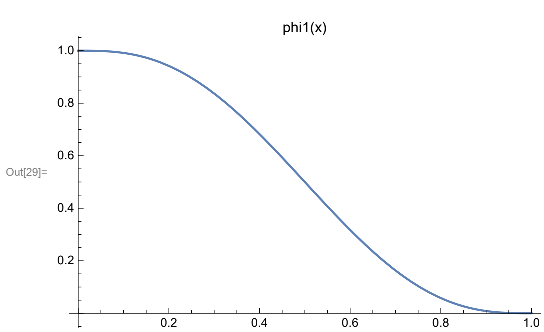

FIGURE 5. $\phi_1(x)$

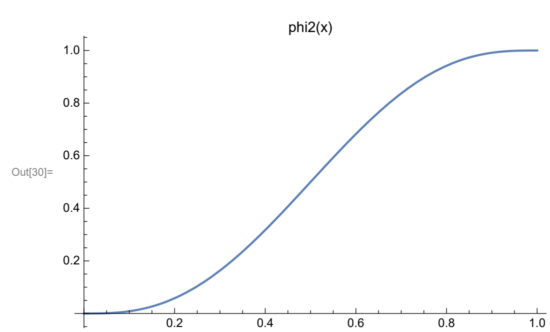

FIGURE 6. $\phi_2(x)$

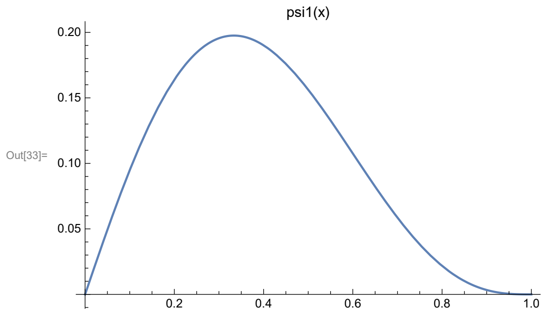

FIGURE 7. $\psi_1(x)$

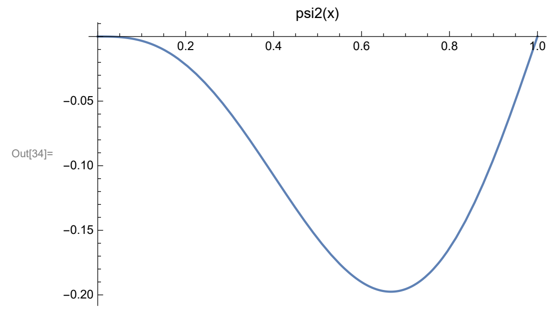

FIGURE 8. $\psi_2(x)$

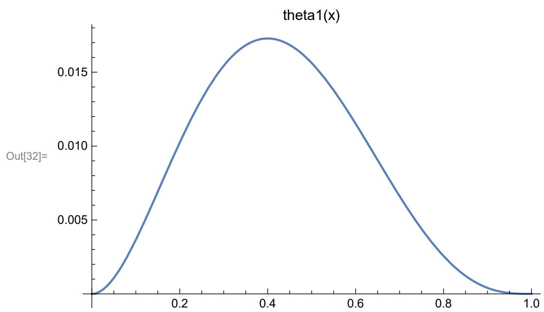

FIGURE 9. $\Theta_1(x)$

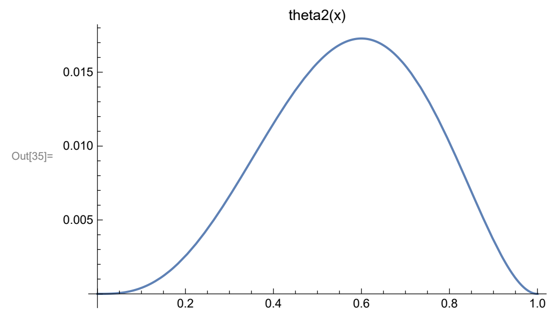

FIGURE 10. $\Theta_2(x)$

And we verify the constraints on these basis functions.

$$
\begin{array}{rrrr}
\phi_1(0) = 1 & \phi_1(1) = 0 & \phi_2(0) = 0 & \phi_2(1) = 1 \\
\phi_1^{(1)}(0) = 0 & \phi_1^{(1)}(1) = 0 & \phi_2^{(1)}(0) = 0 & \phi_2^{(1)}(1) = 0 \\
\phi_1^{(2)}(0) = 0 & \phi_1^{(2)}(1) = 0 & \phi_2^{(2)}(0) = 0 & \phi_2^{(2)}(1) = 0 \\
\psi_1(0) = 0 & \psi_1(1) = 0 & \psi_2(0) = 0 & \psi_2(1) = 0 \\
\psi_1^{(1)}(0) = 1 & \psi_1^{(1)}(1) = 0 & \psi_2^{(1)}(0) = 0 & \psi_2^{(1)}(1) = 1 \\
\psi_1^{(2)}(0) = 0 & \psi_1^{(2)}(1) = 0 & \psi_2^{(2)}(0) = 0 & \psi_2^{(2)}(1) = 0 \\
\Theta_1(0) = 0 & \Theta_1(1) = 0 & \Theta_2(0) = 0 & \Theta_2(1) = 0 \\
\Theta_1^{(1)}(0) = 0 & \Theta_1^{(1)}(1) = 0 & \Theta_2^{(1)}(0) = 0 & \Theta_2^{(1)}(1) = 0 \\
\Theta_1^{(2)}(0) = 1 & \Theta_1^{(2)}(1) = 0 & \Theta_2^{(2)}(0) = 0 & \Theta_2^{(2)}(1) = 1
\end{array}
$$

QUESTION 3

Consider the function:

$$f(x) = -18e^{-10x}(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x})$$

a) Use 2-node, 3-node, all the way to 50-node Lagrange interpolation defined over the entire length $x \in [0, 10]$ to fit the defined polynomial. What is the sum of the error at the nodes $E_{\text{Nodal}} = \sum_{i=1}^{N} |f(x_i) - g(x_i)|$? What is the integral error $E_{\text{Total}} = \int_0^{10} |f(x) - g(x)|dx$?

Hint: To calculate the total error (which is actually L_1 error on the interval), you can simply use large number of points (say 1000), and use $\frac{1}{M}\sum_{k=1}^{M} |f(x_k) - g(x_k)|$ to approximate the integral.

Solution:

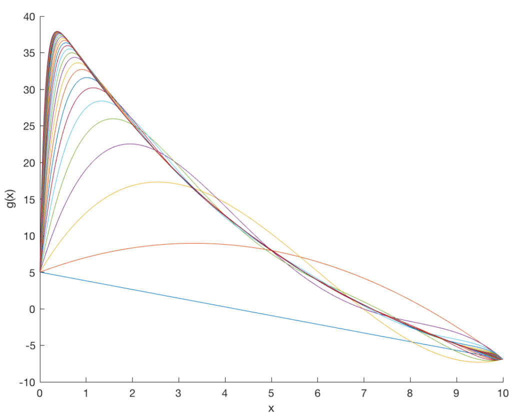

FIGURE 11. Plot of interpolated function with number of nodes from 2 to 50.

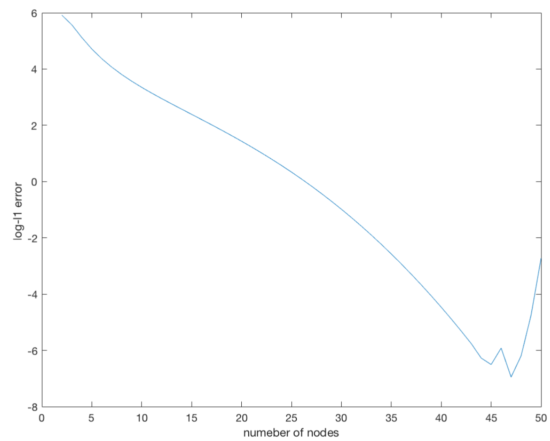

FIGURE 12. Log L1 error vs number of nodes

b) Use piecewise linear Lagrange C_0 interpolation to derive g(x) using between 1 element and 50 elements with the length of each element being constant. Plot your interpolation g(x) against f(x). What are your nodal and integral errors.

Solution:

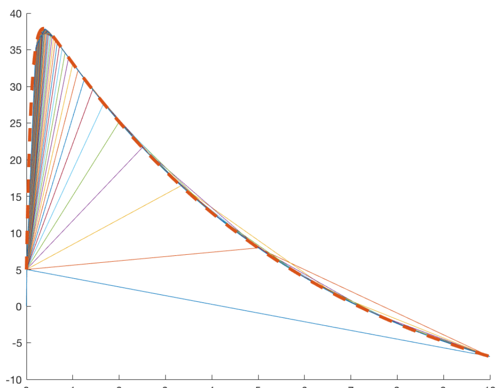

FIGURE 13. Plot of interpolated functions. Number of elements from 1 to 50.

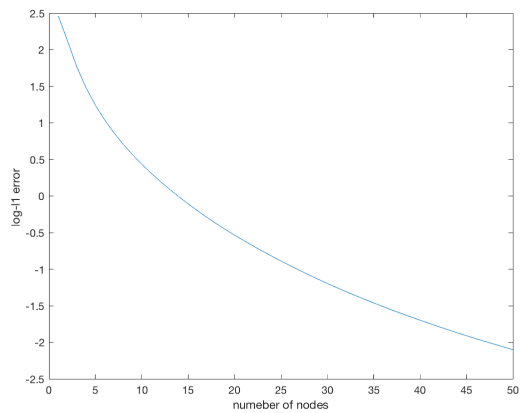

FIGURE 14. *log* L^1 error

c) Repeat part b) but now apply non-equispaced elements (you can decide where to place them) in order to distribute elements and nodes.
Solution:

Since $f(x)$ changes dramatically in the interval $[0, 2]$, and slowly in the interval $[2, 10]$, I picked 80% of the total nodes from the interval $[0, 2]$, and remaining 20% from the interval $[2, 10]$. Applying these non-equispaced elements, the result is a little bit better than the result in part b).

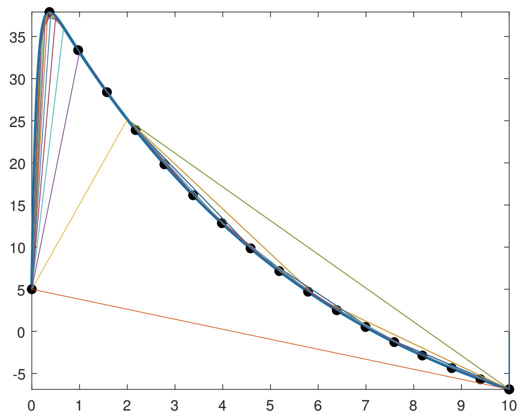

FIGURE 15. Plot of interpolated functions. Number of elements from 1 to 50.

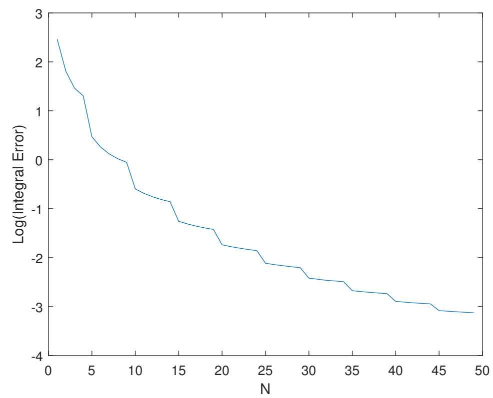

FIGURE 16. *log* L^1 error

QUESTION 4
Consider

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 10f = 3, \qquad x \in [0, 10]$$

with b.c.'s

$$f|_{x=0} = A$$
$$\frac{\mathrm{d}f}{\mathrm{d}x}|_{x=10} = B$$

Assume we will be developing a Galerkin weak weighted residual form and will therefore be using C_0 Lagrange interpolation on three four-node elements.

a) Define all local and global variables.

Solution:

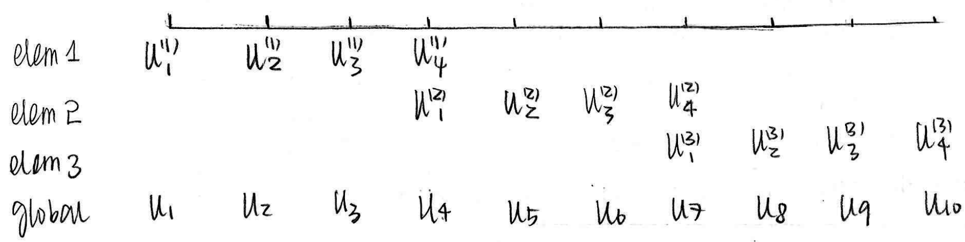

FIGURE 17. Local and Global Nodes

b) Write f_{app} using local expansion over these three elements.

Solution:

Using local expansion, we have

$$\begin{aligned} f_{app} = & u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} + u_3^{(1)}\psi_3^{(1)} + u_4^{(1)}\psi_4^{(1)} + \\ & u_1^{(2)}\psi_1^{(2)} + u_2^{(2)}\psi_2^{(2)} + u_3^{(2)}\psi_3^{(2)} + u_4^{(2)}\psi_4^{(2)} + \\ & u_1^{(3)}\psi_1^{(3)} + u_2^{(3)}\psi_2^{(3)} + u_3^{(3)}\psi_3^{(3)} + u_4^{(3)}\psi_4^{(3)} \end{aligned}$$

c) Sketch the local basis functions.

Solution:

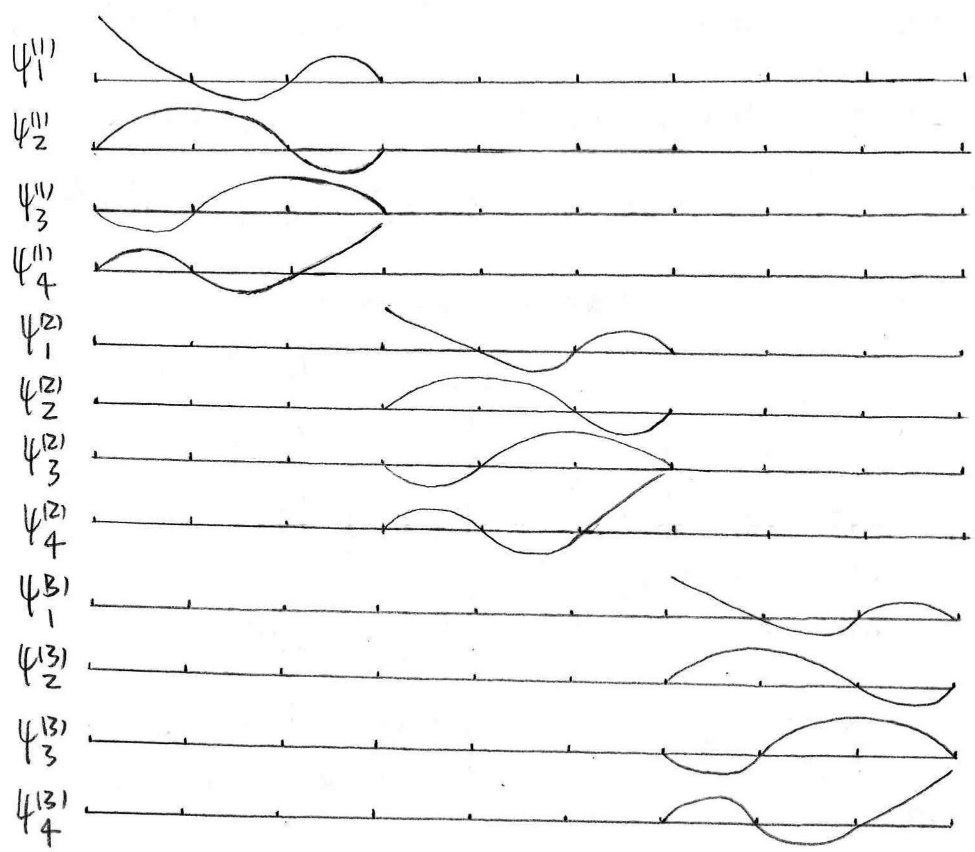

FIGURE 18. Local basis functions

d) Refine f_{app} by incorporating boundary conditions.

Solution:

Since we are using weak form, f_{app} only need to satisfy the essential boundary condition, i.e.

$$f_{app}|_{x=0} = A.$$

From the sketch of all the local basis functions, we find that at the first node $x = 0$, $\psi_1^{(1)}(x = 0) = 1$ while all the other functions are zero at the first node.Thus,

$$f_{app}|_{x=0} = u_1^{(1)} \times 1 = u_1^{(1)} = A$$

e) Map all local variables to global variables and substitute.

Solution:

Enforcing the functional continuity constraints, we have

$$u_4^{(1)} = u_1^{(2)} = u_4$$
$$u_4^{(2)} = u_1^{(3)} = u_7$$

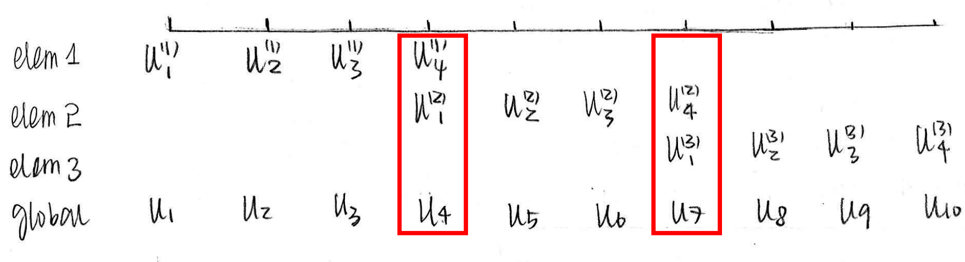

FIGURE 19. Map the local variables to global variables

f) Consolidate the local basis functions into global basis functions.

Solution:

From

$$u_4^{(1)} = u_1^{(2)} = u_4$$
$$u_4^{(2)} = u_1^{(3)} = u_7$$

we have

$$\psi_4 = \psi_4^{(1)} + \psi_1^{(2)}$$
$$\psi_7 = \psi_4^{(2)} + \psi_1^{(3)}$$

And now

$$\begin{aligned}
f_{app} =& u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} + u_3^{(1)}\psi_3^{(1)} + u_4^{(1)}\psi_4^{(1)} + \\
& u_1^{(2)}\psi_1^{(2)} + u_2^{(2)}\psi_2^{(2)} + u_3^{(2)}\psi_3^{(2)} + u_4^{(2)}\psi_4^{(2)} + \\
& u_1^{(3)}\psi_1^{(3)} + u_2^{(3)}\psi_2^{(3)} + u_3^{(3)}\psi_3^{(3)} + u_4^{(3)}\psi_4^{(3)} \\
=& u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} + u_3^{(1)}\psi_3^{(1)} + \\
& u_4^{(1)}(\psi_4^{(1)} + \psi_1^{(2)}) + \\
& u_2^{(2)}\psi_2^{(2)} + u_3^{(2)}\psi_3^{(2)} + \\
& u_4^{(2)}(\psi_4^{(2)} + \psi_1^{(3)}) + \\
& u_2^{(3)}\psi_2^{(3)} + u_3^{(3)}\psi_3^{(3)} + u_4^{(3)}\psi_4^{(3)} \\
=& u_1\psi_1 + u_2\psi_2 + u_3\psi_3 + u_4\psi_4 + u_5\psi_5 + \\
& u_6\psi_6 + u_7\psi_7 + u_8\psi_8 + u_9\psi_9 + u_{10}\psi_{10}
\end{aligned}$$

g) Sketch global basis functions.

Solution:

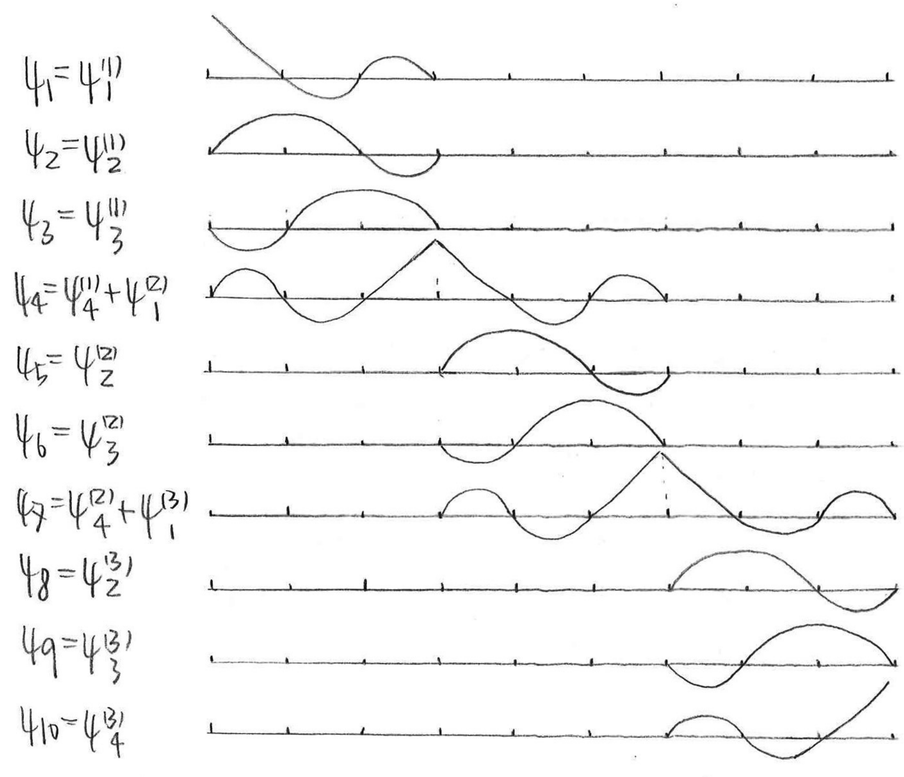

FIGURE 20. Global basis functions

h) Identify how many local unknowns, global unknowns, and functional continuity constraints are.

Solution:

There are in total 11 local unknowns, 9 global unknowns, and 2 functional constraints. (From the essential boundary condition, we already have $u_1 = u_1^{(1)} = A$.)

i) How many weighting equations will you need implement globally.

Solution:

We will need 9 weighting equations to implement globally.

QUESTION 5 Repeat Problem 4 except now assume that we will be developing a least square weighted residual form and will therefore be using C_1 Hermite interpolation.

a) Define all local and global variables.

Solution:

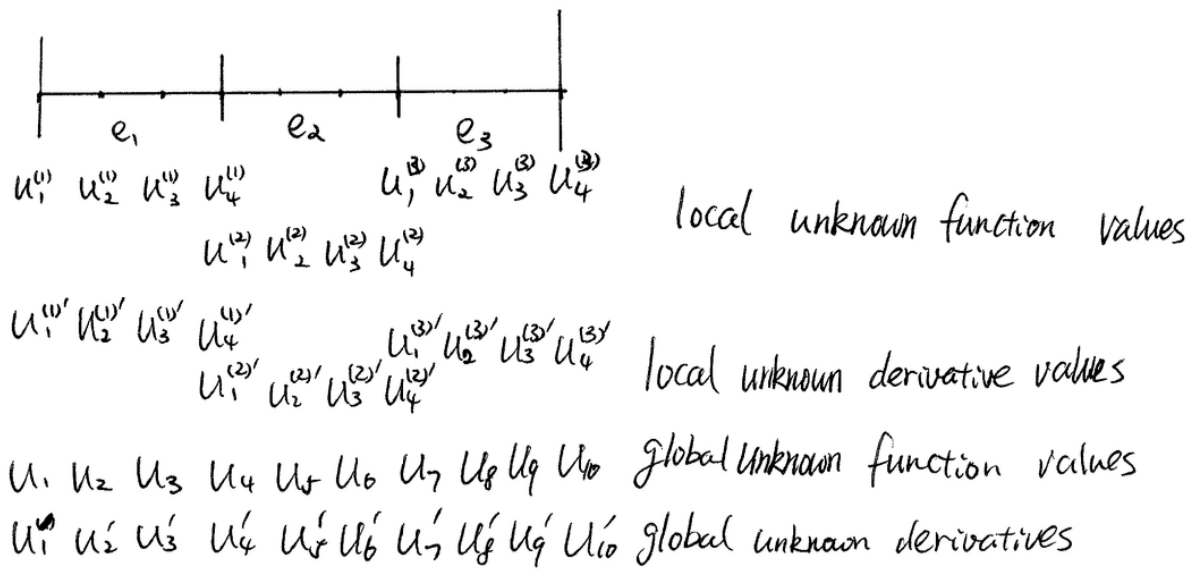

FIGURE 21. Local and Global Nodes

b) Write f_{app} using local expansion over these three elements.

Solution:

Using local expansion, we have

$$\begin{aligned} f_{app} =& u_1^{(1)}\phi_1^{(1)} + u_2^{(1)}\phi_2^{(1)} + u_3^{(1)}\phi_3^{(1)} + u_4^{(1)}\phi_4^{(1)} + u_1^{(1)'}\psi_1^{(1)} + u_2^{(1)'}\psi_2^{(1)} + u_3^{(1)'}\psi_3^{(1)} + u_4^{(1)'}\psi_4^{(1)} + \\ & u_1^{(2)}\phi_1^{(2)} + u_2^{(2)}\phi_2^{(2)} + u_3^{(2)}\phi_3^{(2)} + u_4^{(2)}\phi_4^{(2)} + u_1^{(2)'}\psi_1^{(2)} + u_2^{(2)'}\psi_2^{(2)} + u_3^{(2)'}\psi_3^{(2)} + u_4^{(2)}\psi_4^{(2)} + \\ & u_1^{(3)}\phi_1^{(3)} + u_2^{(3)}\phi_2^{(3)} + u_3^{(3)}\phi_3^{(3)} + u_4^{(3)}\phi_4^{(3)} + u_1^{(3)'}\psi_1^{(3)} + u_2^{(3)'}\psi_2^{(3)} + u_3^{(3)'}\psi_3^{(3)} + u_4^{(3)}\psi_4^{(3)}. \end{aligned}$$

c) Sketch the local basis functions.

Solution:

Please see the following page.

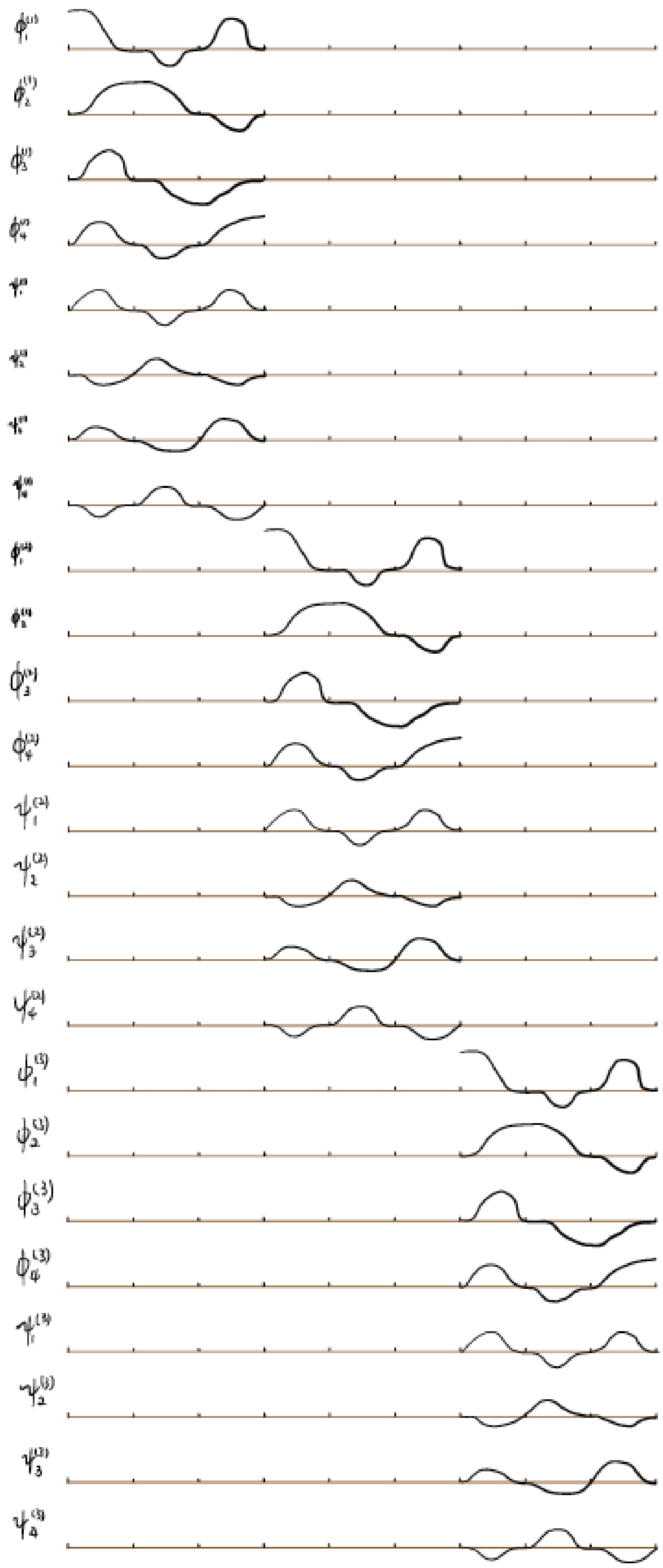

FIGURE 22. Local basis functions

d) Refine f_{app} by incorporating boundary conditions.

Solution:

Now we are using least square weighted residual form, f_{app} should satisfy not only the essential boundary conditions, but also natural boundary conditions. Therefore, we have

$$f_{app}|_{x=0} = A$$
$$\frac{\mathrm{d}f_{app}}{\mathrm{d}x}|_{x=10} = B$$

At the first node, i.e., $x = 0$, only $\phi_1^{(1)}(x = 0) = 1$, all other Hermite basis functions are defined as zero at the first node.
Similarly, at the last node, i.e., $x = 10$, only $\frac{\mathrm{d}\psi_4^{(3)}}{\mathrm{d}x}|_{x=10} = 1$, all other Hermite basis functions have zero slope at the last node.

Thus,

$$u_1^{(1)} = A$$
$$u_4^{(3)'} = B$$

e) Map all local variables to global variables and substitute.

Solution:

Enforcing the functional continuity constraints, we have

$$u_4^{(1)} = u_1^{(2)} = u_4$$
$$u_4^{(2)} = u_1^{(3)} = u_7$$

Similarly, enforcing the first-derivative continuity constraints, we have

$$u_4^{(1)'} = u_1^{(2)'} = u_4^{'}$$
$$u_4^{(2)'} = u_1^{(3)'} = u_7^{'}$$

For i = 1,2,3 and j = 2,3, we can map the variables $u_j^{(i)}$ and $u_j^{(i)'}$ as follow:

$$u_2 = u_2^{(1)}$$
$$u_3 = u_3^{(1)}$$
$$u_5 = u_2^{(2)}$$
$$u_6 = u_3^{(2)}$$
$$u_8 = u_2^{(3)}$$
$$u_9 = u_3^{(3)}$$
$$u_2' = u_2^{(1)'}$$
$$u_3' = u_3^{(1)'}$$
$$u_5' = u_2^{(2)'}$$
$$u_6' = u_3^{(2)'}$$
$$u_8' = u_2^{(3)'}$$
$$u_9' = u_3^{(3)'}$$

f) Consolidate the local basis functions into global basis functions.

Solution:

From

$$u_4^{(1)} = u_1^{(2)} = u_4$$
$$u_4^{(2)} = u_1^{(3)} = u_7$$
$$u_4^{(1)'} = u_1^{(2)'} = u_4^{'}$$
$$u_4^{(2)'} = u_1^{(3)'} = u_7^{'}$$

we have

$$\phi_1 = \phi_1^{(1)}$$
$$\phi_2 = \phi_2^{(1)}$$
$$\phi_3 = \phi_3^{(1)}$$
$$\phi_4 = \phi_4^{(1)} + \phi_1^{(2)}$$
$$\phi_5 = \phi_1^{(2)}$$
$$\phi_6 = \phi_2^{(2)}$$
$$\phi_7 = \phi_4^{(2)} + \phi_1^{(3)}$$
$$\phi_8 = \phi_2^{(3)}$$
$$\phi_9 = \phi_3^{(3)}$$
$$\phi_{10} = \phi_4^{(3)}$$
$$\psi_1 = \psi_1^{(1)}$$
$$\psi_2 = \psi_2^{(1)}$$
$$\psi_3 = \psi_3^{(1)}$$
$$\psi_4 = \psi_4^{(1)} + \psi_1^{(2)}$$
$$\psi_5 = \psi_1^{(2)}$$
$$\psi_6 = \psi_2^{(2)}$$
$$\psi_7 = \psi_4^{(2)} + \psi_1^{(3)}$$
$$\psi_8 = \psi_2^{(3)}$$
$$\psi_9 = \psi_3^{(3)}$$
$$\psi_{10} = \psi_4^{(3)}$$

Using global expansion, f_{app} can be expressed into

$$f_{app} = \sum_{i=1}^{10} (u_i \phi_i + u_i' \psi_i)$$

g) Sketch global basis functions.

Solution:

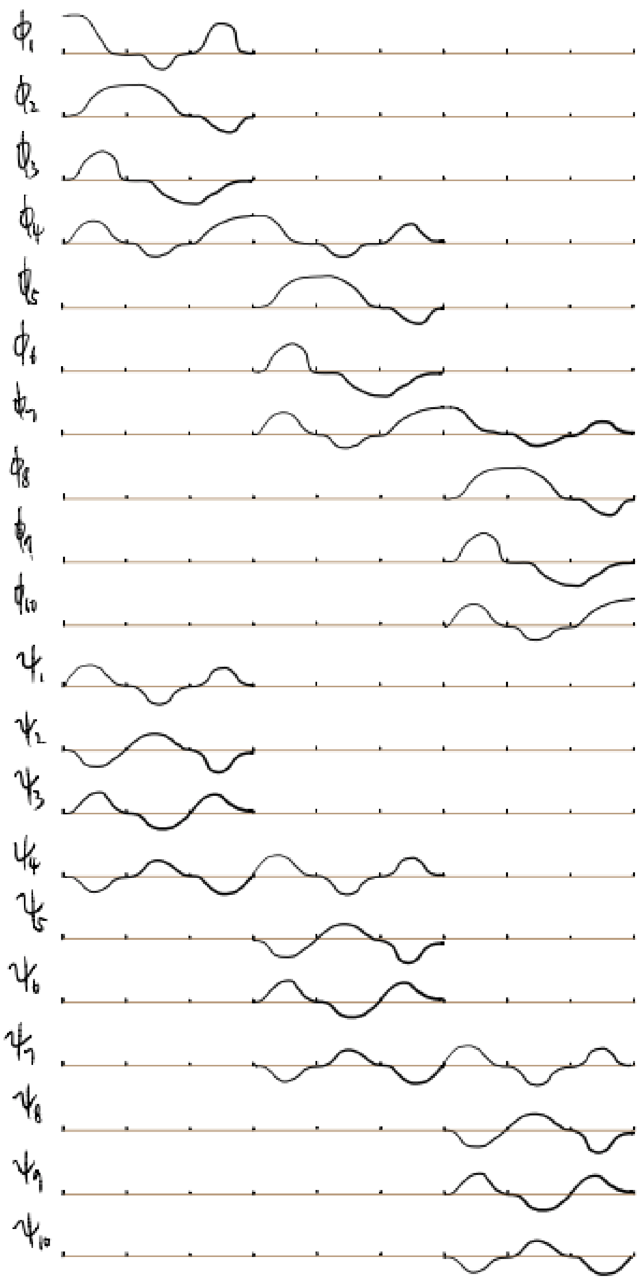

Figure 23. Global basis functions

h) Identify how many local unknowns, global unknowns, and functional continuity constraints are.

Solution:

There are in total 24 local unknowns, 20 global unknowns, 2 functional constraints, 2 derivative constraints.

i) How many weighting equations will you need implement globally.

Solution:

We will need 20 weighting equations to implement globally.