UNIVERSITY OF NOTRE DAME
Department of Civil and Environmental Engineering and Earth Sciences

CE 60130 Finite Elements in Engineering — **February 1^{st}, 2018**
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Homework Set #2

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$\frac{1}{10}\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5 \qquad \text{and} \qquad \frac{du}{dx}_{|x=10} = \frac{-3\pi}{5}$$

Its analytical solution is given by:

$$u(x) = -18e^{-10x}\left(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x}\right)$$

We will examine components of solving this ODE with the method of weighted residuals using different kind of basis functions. Recall that the approximate solution follows

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

where u_B is the boundary function that satisfies the boundary conditions, α_i are the unknown coefficients which must be determined and $\phi_i(x)$ are the known basis functions that form a complete set.

QUESTION 1

a) *Formulate the boundary function u_B such that the boundary conditions are satisfied. Hint: The boundary function must satisfy both derivative and functional value boundary conditions.*

b) *Let us examine using a polynomial expansion for this PDE. Show that the basis functions*

$$\phi_i(x) = x^i\left(\frac{10(i+1)}{i} - x\right)$$

satisfy the admissibility and completeness criteria.

c) *For the given differential equation, $\phi_i(x)$, your selection of u_B, implement the Galerkin residual method, where $w_j(x) = \phi_j(x)$. Express the generic term of the matrix $\langle \mathcal{L}(\phi_i(x)), \phi_j(x)\rangle_\Omega$ as a function of i and j and the generic term of the vector at the right hand side as a function of j.*

d) *Develop a code to solve each system using the Galerkin weight method. Try different values of N and compute the condition number C of the matrix. Try different values of N starting with $N = 1$ increasing it until you cannot get a solution. Create a plot N v/s $log(cond())$ and plot the different u_{app} solution for each expansion set and the analytical solution in the same figure. Comment on the results as you increase N. Does any of your solutions fit the analytical solution? If not, explain why you think it doesn't converge.*

e) *Determine the L_2 error by comparing with the point based analytical solution. Plot N versus $log(\mathcal{E})$. The L_2 error can be calculated as the Root Mean Square Error, as follows:*

$$\mathcal{E} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - f_i)^2}$$

where N is the number of points in the point-wise comparison, f_i is the exact value at the i^{th} point and y_i is the value given by your model at the i^{th} point.

QUESTION 2

Considering the ODE in the previous Question, now let us examine using sine/cosine expansion for this PDE. Repeat ***Question 1 b), c), d), and e)***

$$\phi_i(x) = \sin\left(\frac{1}{20}\pi(2i-1)x\right)$$

QUESTION 3

Comment on your solution of Question 1 and 2. Which set of basis function works better?. Explain why one set of basis functions works better than the other. Does it have anything to do with the condition number of each matrix?