**UNIVERSITY OF NOTRE DAME**
**Department of Civil and Environmental Engineering**
**and Earth Sciences**

**CE 60130 Finite Elements in Engineering** **March $8^{th}$, 2018**
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# Homework Set #5

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$D\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5 \qquad \text{and} \qquad \left[D\frac{du}{dx} + u\right]_{x=10} = -7.0705084$$

Its analytical solution is given by:

$$u(x) = -18e^{-10x}\left(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x}\right) \qquad \text{when } D = 0.1$$

$$u(x) = -18e^{-x}\left(5.09738e^{0.211325x} - 6.37516e^{0.788675x} + e^{x}\right) \qquad \text{when } D = 1$$

Develop the symmetrical weak weighted residual form of the Galerkin method to solve this ODE on the domain $0 \le x \le 10$ using the finite element method with $C_0$ Lagrange basis functions.

**QUESTION 1:** Assume that your approximate solution is designated as $\hat{u}$.

a) *Formulate the interior domain error.*

b) *Formulate the natural boundary error.*

c) *Formulate the fundamental weak weighed residual form using integration by parts to determine the symmetrical weak form of the Galerkin method, where $w_j(x) = \phi_j(x)$. The combined interior error and the natural boundary error must go to zero, that is:*

$$< \epsilon_I, w_j >_\omega + < \epsilon_{B,N}, w_j >_{B,N} = 0 \quad j = 1, .., N.$$

d) *Using either index or vector notation, develop the elemental matrices and elemental right hand side vector for any order Lagrange $C_0$ interpolation.*

e) *For a generic linear element of length $\Delta x_j$, develop through analytical integration the elemental matrices and right hand side load vector.*

f) *Consider the grid in Figure 1. Load the elemental matrices and vector into a global system.*

g) *Implement the essential boundary condition.*

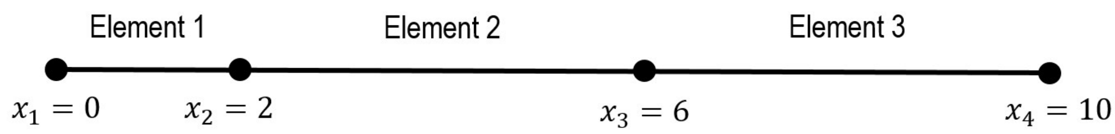


FIGURE 1. Figure for question 1. f)

h) *Solve the 4x4 system for $D = 0.1$ and $D = 1$ and plot both your solution and the analytical solution*

**QUESTION 2**

a) *Now consider $N$ equally spaced elements of length $10/N$ and create a code to solve a $(N+1)$ by $(N+1)$ system following the steps that you used in the previous question. Plot your solution for $D = 0.1$ with $N = 5$, $N = 25$, $N = 75$, and $N = 150$ along with the analytical solution.*

b) *Repeat a), but now consider $D = 1$.*

c) *For both $N = 5$ and $N = 150$, comment on the difference of using $D = 0.1$ and $D = 1$, and relate it to the structure of the matrix.*