UNIVERSITY OF NOTRE DAME
Department of Civil and Environmental Engineering
and Earth Sciences

CE 60130 Finite Elements in Engineering **February 15^{th}, 2018**
J.J. Westerink **Due: February 22^{nd}, 2018**

Homework Set #3

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$\frac{1}{10}\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5 \qquad \text{and} \qquad \frac{1}{10}\frac{du}{dx}|_{x=10} = \frac{-3\pi}{50}$$

Its analytical solution is given by:

$$u(x) = -18e^{-10x}\left(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x}\right)$$

Develop a code using **the symmetrical weak weighted residual form of the Galerkin method** to solve this ODE on the domain $0 \leq x \leq 10$ using polynomial basis functions. Recall that the approximate solution follows

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

where u_B is the boundary function that satisfies the **essential** boundary conditions, α_i are the unknown coefficients which must be determined and $\phi_i(x)$ are the known basis functions that form a complete set.

QUESTION 1

a) *Determine the essential and natural boundary conditions.*

b) *Formulate the boundary function u_B. Hint: The boundary function must satisfy only the essential boundary condition.*

c) *Let us examine using a polynomial expansion for this ODE. Show that the basis functions*

$$\phi_i(x) = x^i$$

satisfy the admissibility and completeness criteria.

d) *For the given differential equation, $\phi_i(x)$, your selection of u_B, using integration by parts to determine the symmetrical weak form of the Galerkin method, where $w_j(x) = \phi_j(x)$. Express the generic term of the matrix as a function of i and j and the generic term of the vector at the right hand side as a function of j.* Hint: Both the interior error and the natural boundary error must go to zero, that is

$$< \epsilon_I, w_j >_\Omega + < \epsilon_{B,N}, w_j >_{B,N} = 0, \qquad j = 1, ..., N.$$

And we can apply integration by parts once on the second-derivative term, so that some boundary terms can be cancelled out.

e) *Develop a code to solve each system using the symmetrical weak weighted residual Galerkin method. Try different values of N and compute the condition number C of the matrix. Try different values of N starting with $N = 1$ increasing it until $N = 150$. Create a plot N v/s log(cond()) and plot the different u_{app} solution for each expansion set and the analytical solution in the same figure. Comment on the results as you increase N. Does any of your solutions fit the analytical solution? If not, explain why you think it doesn't converge.*

f) *Determine the L_2 error by comparing with the point based analytical solution. Plot N versus log($\mathcal{E}$). The L_2 error can be calculated as the Root Mean Square Error, as follows:*

$$\mathcal{E} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(y_i - f_i)^2},$$

where M is the number of points in the point-wise comparison, f_i is the exact value at the i^{th} point and y_i is the value given by your model at the i^{th} point.