UNIVERSITY OF NOTRE DAME
Department of Civil and Environmental Engineering and Earth Sciences

CE 60130 Finite Elements in Engineering — **February 15th, 2018**
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Homework Set #3 KEY

Consider the following ordinary differential equation (ODE) of $u(x)$:

$$\frac{1}{10}\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{u}{6} = -3, \qquad x \in [0, 10]$$

with boundary conditions

$$u|_{x=0} = 5 \qquad \text{and} \qquad \frac{1}{10}\frac{du}{dx}|_{x=10} = \frac{-3\pi}{50}$$

Its analytical solution is given by:

$$u(x) = -18e^{-10x}\left(2.08781e^{0.169541x} - 3.36559e^{9.83046x} + e^{10x}\right)$$

Develop a code using **the symmetrical weak weighted residual form of the Galerkin method** to solve this ODE on the domain $0 \leq x \leq 10$ using polynomial basis functions. Recall that the approximate solution follows

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x)$$

where u_B is the boundary function that satisfies the **essential** boundary conditions, α_i are the unknown coefficients which must be determined and $\phi_i(x)$ are the known basis functions that form a complete set.

QUESTION 1

a) *Determine the essential and natural boundary conditions.*

Solution: Recall the definition in the slide, essential and natural boundary condition are defined as following:

$$< v, L(u) >=< u, L^*(v) > + \int_\Gamma (F(v)G(u) - F(u)G^*(v))dS$$

where F and G defines the essential and natural boundary condition respectively.

Let v be arbitrary test function and we proceed with the procedure of integration by parts.

$$\begin{aligned}
< Lu, v > &= \frac{1}{10} < \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}, v > + < \frac{\mathrm{d}u}{\mathrm{d}x}, v > + \frac{1}{6} < u, v > \\
&= \frac{1}{10}\left[\frac{\mathrm{d}u}{\mathrm{d}x}v\right]_0^{10} - \frac{1}{10}\int_0^{10} \frac{\mathrm{d}u}{\mathrm{d}x} \cdot \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + [uv]_0^{10} - \int_0^{10} u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int_0^{10} uv\,\mathrm{d}x \\
&= \int_0^{10} u(\frac{1}{10}\frac{\mathrm{d}^2 v}{x^2} - \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{v}{6})\,\mathrm{d}x + \left[\frac{1}{10}v\frac{\mathrm{d}u}{\mathrm{d}x} + uv - \frac{1}{10}u\frac{\mathrm{d}v}{\mathrm{d}x}\right]_0^{10}
\end{aligned}$$

Therefore, we can conclude that

$$F(v) = v \qquad\qquad F(u) = u$$

$$G(u) = \frac{1}{10}\frac{\mathrm{d}u}{\mathrm{d}x} \qquad\qquad G^*(v) = \frac{1}{10}\frac{\mathrm{d}v}{\mathrm{d}x} - v$$

And, hence, F defines the essential boundary condition and G and G^* define the natural boundary condition.

b) *Formulate the boundary function u_B. Hint: The boundary function must satisfy only the essential boundary condition.*

Solution:

In the fundamental weak form, the boundary function u_B only needs to satisfy the essential boundary condition, i.e.

$$u_B|_{x=0} = 0.$$

To satisfy the above condition, we can easily choose $u_B = 5$.

c) *Let us examine using a polynomial expansion for this ODE. Show that the basis functions*

$$\phi_i(x) = x^i$$

satisfy the admissibility and completeness criteria.

Solution:

For admissibility we need $u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i$ to satisfy the boundary conditions and the correct degree of functional continuity. So first we need ϕ_i to satisfy the homogeneous form of the essential boundary conditions, that is, for all i:

$$\phi_i|_{x=0} = 0.$$

Let: $\phi_i = x^i$. Then:

$$\phi_i(x = 0) = 0 \quad \forall i \in \mathcal{N}$$

Since our ODE is second order, for functional continuity we need

$$u_{app} \in W^{(2)} = C_1.$$

Since $\phi_i(x) \in W^{(\infty)}$, u_{app} satisfies the correct degree of functional continuity.

For completeness, we need the ϕ_i's to form a linearly independent complete set, which means that as we add more and more terms, $u_{app} \to u$. This set of polynomial functions satisfies this requirement.

d) *For the given differential equation, $\phi_i(x)$, your selection of u_B, using integration by parts to determine the symmetrical weak form of the Galerkin method, where $w_j(x) = \phi_j(x)$. Express the generic term of the matrix as a function of i and j and the generic term of the vector at the right hand side as a function of j.* Hint: Both the interior error and the natural boundary error must go to zero, that is

$$< \epsilon_I, w_j >_\Omega + < \epsilon_{B,N}, w_j >_{B,N} = 0, \qquad j = 1, ..., N.$$

And we can apply integration by parts once on the second-derivative term, so that some boundary condition terms can be cancelled out.

Solution:

Since we are not satisfying the natural boundary condition, we also consider the error associated with the locations where the natural boundary condition is violated. Therefore,

$$\text{The interior error:} \quad \epsilon_I = \mathcal{L}(u_{app}) - p(x) = \mathcal{L}(u_B) + \sum_{i=1}^{N} \alpha_i \mathcal{L}(\phi_i) - p(x)$$

$$\text{The natural boundary error:} \quad \epsilon_{B,N} = -\mathcal{S}(u_{app}) + g_N(x) = -\sum_{i=1}^{N} \alpha_i \mathcal{S}(\phi_i) + g_N(x)$$

Galerkin method implies that $\phi_j = w_j$. Thus, we can substitute the w_j as follows:

$$< \epsilon_I, w_j >_\Omega + < \epsilon_{B,N}, w_j >_{B,N} = 0, \qquad j = 1, ..., N$$

that is,

$$\int_\Omega [\mathcal{L}(u_B) + \sum_{i=1}^{N} \alpha_i \mathcal{L}(\phi_i) - p(x)]\phi_j \, \mathrm{d}x + [(-\sum_{i=1}^{N} \alpha_i \mathcal{S}(\phi_i) + g_N(x))\phi_j]|_{B,N} = 0, \quad j = 1, ..., N$$

$$\int_\Omega \mathcal{L}(u_B)\phi_j \, \mathrm{d}x + \sum_{i=1}^{N} \alpha_i \int_\Omega (\frac{1}{10}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2}\phi_j + \frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_j + \frac{1}{6}\phi_i\phi_j) \, \mathrm{d}x - \int_\Omega p\phi_j \, \mathrm{d}x - \sum_{i=1}^{N} \alpha_i (\frac{1}{10}\frac{\mathrm{d}\phi_i}{\mathrm{d}x})|_{x=10} + (g\phi_j)|_{x=10} = 0$$

Applying integration by parts on the second-derivative term, we get

$$\sum_{i=1}^{N} \alpha_i [\int_\Omega (\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_j + \frac{1}{6}\phi_i\phi_j) \, \mathrm{d}x + (\frac{1}{10}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_j)|_0^{10} - \int_\Omega \frac{1}{10}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_j}{\mathrm{d}x} \, \mathrm{d}x - (\frac{1}{10}\frac{\mathrm{d}\phi_i}{\mathrm{d}x})|_{x=10}]$$
$$= -\int_\Omega \mathcal{L}(u_B)\phi_j \, \mathrm{d}x + \int_\Omega p\phi_j \, \mathrm{d}x - (g\phi_j)|_{x=10}$$

Thus, the generic terms are:

$$\begin{aligned} a_{ji} &= \int_\Omega (\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\phi_j + \frac{1}{6}\phi_i\phi_j - \frac{1}{10}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\frac{\mathrm{d}\phi_j}{\mathrm{d}x}) \, \mathrm{d}x \\ &= \int_0^{10} i x^{i-1} x^j \, \mathrm{d}x + \int_0^{10} \frac{1}{6} x^i x^j \, \mathrm{d}x - \int_0^{10} \frac{1}{10} i x^{i-1} j x^{j-1} \, \mathrm{d}x \\ &= \frac{i}{i+j} 10^{i+j} + \frac{1}{6}\frac{1}{i+j+1} 10^{i+j+1} - \frac{1}{10}\frac{ij}{i+j-1} 10^{i+j-1} \end{aligned}$$

$$
\begin{aligned}
b_j &= -\int_\Omega \mathcal{L}(u_B)\phi_j \,\mathrm{d}x + \int_\Omega p\phi_j \,\mathrm{d}x - (g\phi_j)|_{x=10} \\
&= -\int_0^{10} \frac{5}{6} x^j \,\mathrm{d}x + \int_0^{10} (-3) x^j \,\mathrm{d}x + \frac{3\pi}{50} 10^j \\
&= -\frac{23}{6} \frac{1}{j+1} 10^{j+1} + \frac{3\pi}{50} 10^j
\end{aligned}
$$

e) *Develop a code to solve each system using the fundamental weak weighted residual Galerkin method. Try different values of N and compute the condition number C of the matrix. Try different values of N starting with $N = 1$ increasing it until $N = 150$. Create a plot N v/s log(cond()) and plot the different u_{app} solution for each expansion set and the analytical solution in the same figure. Comment on the results as you increase N. Does any of your solutions fit the analytical solution? If not, explain why you think it doesn't converge.*

Solution:

See Figures 1 and 2. cond() keeps increasing, u_{app} gets closer to the analytical solution.

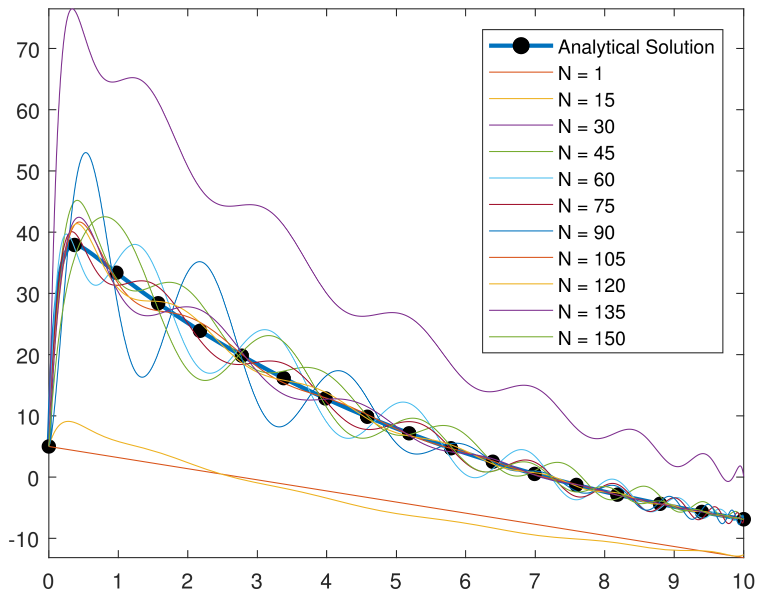

FIGURE 1. Polynomial expansions for different values of N

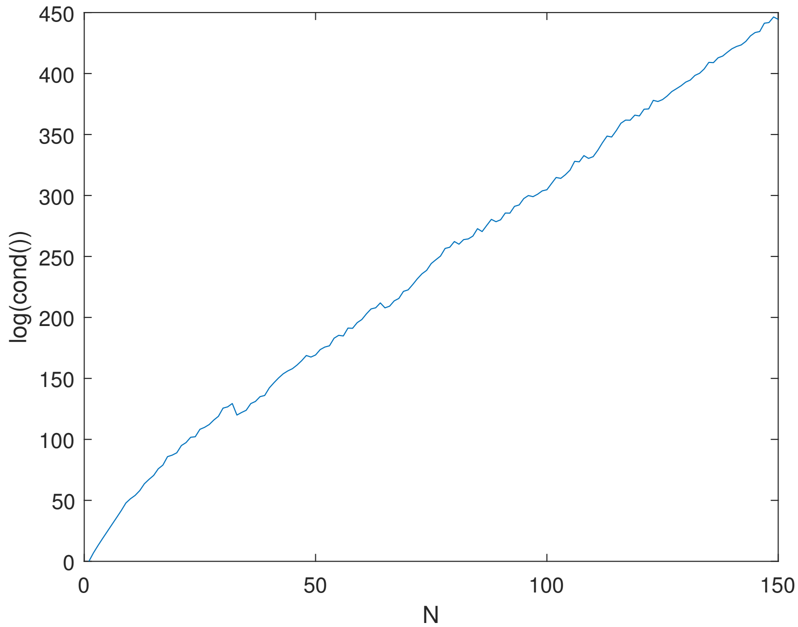

FIGURE 2. $\log C$ v/s N

f) *Determine the L_2 error by comparing with the point based analytical solution. Plot N versus $log(\mathcal{E})$. The L_2 error can be calculated as the Root Mean Square Error, as follows:*

$$\mathcal{E} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(y_i - f_i)^2},$$

where M is the number of points in the point-wise comparison, f_i is the exact value at the i^{th} point and y_i is the value given by your model at the i^{th} point.

Solution:
See Figure 3, there is a clear trend that the error is decreasing.

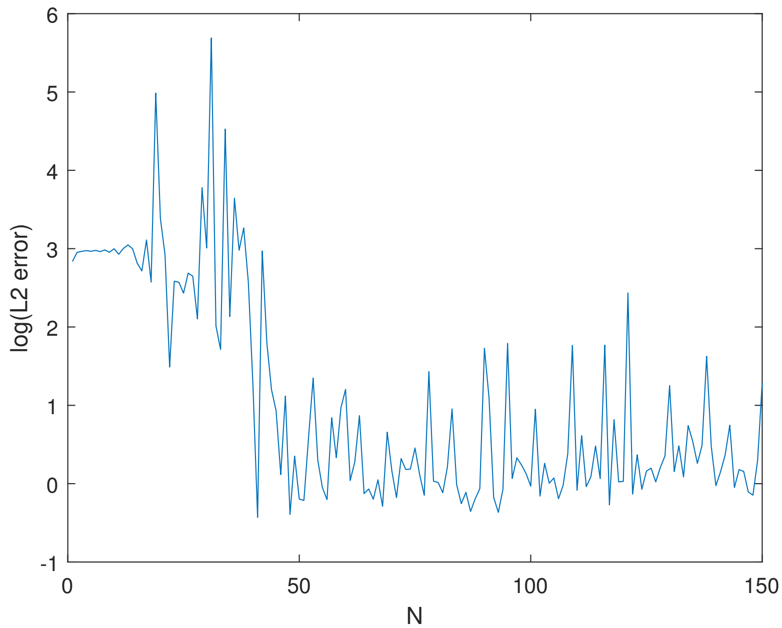

FIGURE 3. $\log L_2$ v/s N