Lecture No. 6

Example 2-D Problem

- Potential flow problems: Seepage in a granular soil. Assume that the soil is isotropic and Darcy’s law gives:

\[ \nu_x = K \frac{\partial u}{\partial x} \quad \text{and} \quad \nu_y = K \frac{\partial u}{\partial y} \]

where

\[ K = \text{permeability coefficient} \]
\[ u = \text{head} \]
\[ \nu_x \text{ and } \nu_y \text{ are velocities in the } x \text{ and } y \text{ directions} \]

- Continuity dictates that

\[ \frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} = 0 \]

- Substituting for \( \nu_x \) and \( \nu_y \) we have

\[ L(u) = K \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \]
• Let’s first check what the essential and natural b.c.’s are

\[
\langle L(u), w \rangle = \iint_{\Omega} K(u_{xx} + u_{yy})w \, dx \, dy
\]

• Integrate by parts using Green’s theorem

\[
\iint_{\Omega} f \cdot g_x \, dx \, dy = \int_{\Gamma} f \cdot g \cos(n, x) \, d\Gamma - \iint_{\Omega} f_xg \, dx \, dy
\]

Thus we integrate both terms by parts to obtain

\[
\iint_{\Omega} K(u_{xx} + u_{yy})w \, dx \, dy = \int_{\Gamma} K[u \alpha_{nx} + u_y \alpha_{ny}] \, w \, d\Gamma
\]

\[
- \iint_{\Omega} [u_x w_x + u_y w_y] \, d\Omega
\]
We note that

\[ u_x \alpha_{nx} + u_y \alpha_{ny} = \frac{\partial u}{\partial x} \frac{\partial \partial}{\partial n} + \frac{\partial u}{\partial y} \frac{\partial \partial}{\partial n} = \frac{\partial u}{\partial n} \]

where \( \frac{\partial x}{\partial n} = \alpha_{nx}, \frac{\partial y}{\partial n} = \alpha_{ny} \) and \( n \) is the unit normal to the boundary.

We note that the boundary term specifically has the following meaning

\[ v_x = Ku_x \text{ and } v_y = Ku_y \]

thus

\[ K(u_x \alpha_{nx} + u_y \alpha_{ny}) = v_x \alpha_{nx} + v_y \alpha_{ny} \]

\[ = v_n \]

\[ = \text{normal velocity} \left( K \frac{\partial u}{\partial n} \right) \]

Thus

\[ \iint_{\Omega} K(u_{xx} + u_{yy})w \partial x \partial y = \int_{\Gamma} K \frac{\partial u}{\partial n} w \partial \partial - \iint_{\Omega} K[u_xw_x + u_yw_y] \partial \partial \]
Thus the b.c.’s are as follows:

- Essential b.c.’s
  \[ u = \bar{u} \text{ on } \Gamma_E \] (specify potential head) on \( \Gamma_B \)

- Natural b.c.’s
  \[ K \frac{\partial u}{\partial n} = \bar{v}_n \text{ on } \Gamma_N \] (specify normal velocity)

Let’s now set up the fundamental weak form:

\[
\langle \mathcal{E}_I, w_j \rangle_\Omega + \langle \mathcal{E}_{B,N}, w_j \rangle_{\Gamma_N} = 0 \quad j = 1, N
\]

\[
\Rightarrow \int_{\Omega} \int_{\Gamma_N} K \left[ u_{,xx} + u_{,yy} \right] w_j \, dx \, dy + \int_{\Gamma_N} \left[ \bar{v}_n - K \frac{\partial u}{\partial n} \right] w_j \, d\Gamma = 0 \quad j = 1, N
\]

Let’s now integrate by parts:

\[
- \int_{\Omega} \int K \left[ u_{,x} w_{j,x} + u_{,y} w_{j,y} \right] d\Omega + \int_{\Gamma} K \frac{\partial u}{\partial n} w_j d\Gamma + \int_{\Gamma_N} \left[ \bar{v}_n - K \frac{\partial u}{\partial n} \right] w_j d\Gamma = 0 \quad j = 1, N
\]
• Now we expand the boundary terms:

\[-\iint_{\Omega} K \left[ u_x w_j, x + u_y w_j, y \right] d\Omega + \int_{\Gamma_N} K \frac{\partial u}{\partial n} w_j d\Gamma + \int_{\Gamma_E} K \frac{\partial u}{\partial n} w_j d\Gamma + \int_{\Gamma_N} \nu_n w_j d\Gamma - \int_{\Gamma_N} K \frac{\partial u}{\partial n} w_j d\Gamma = 0\]

Furthermore we note that \( w_j = 0 \) on \( \Gamma_E \) since all \( \phi_i (= w_j) \) must satisfy the essential b.c.’s

• Thus the symmetrical weak form is:

\[-\iint_{\Omega} K \left( u_x w_j, x + u_y w_j, y \right) d\Omega + \int_{\Gamma_N} \nu_n w_j d\Gamma = 0 \quad j = 1, N\]

• The space requirements for both \( \phi_i \) and \( w_i \) are the same, \( W^{(1)} = C_o \)

• We only require the essential b.c.’s to be satisfied on \( \Gamma_E \)
Time Dependent Problems

1. Use FD’s for the time dependence and Galerkin for the spatial dependence. Applying the Galerkin methods in space leads to sets of time dependent o.d.e.’s. We resolve the time dependence in these equations with finite difference (FD) methods.

The time differencing scheme selected can control the success or failure of the method.

\[ u_{app} = u_B + \sum_{i=1}^{N} \alpha_i(t)\phi_i(x) \]

The coefficients on \( \alpha_i(t) \) are now time dependent.

2. Direct use of the Galerkin Approach in space and time:

\[ u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i(x, t) \]

The interpolating functions, \( \phi_i(x, t) \) are now functions in time as well as space.
Example
Let’s look at the wave equation:

\[ \lambda \nabla^2 u = \frac{\partial^2 u}{\partial t^2} \text{ in } \Omega \]

\[ \Rightarrow \]

\[ \lambda \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \frac{\partial^2 u}{\partial t^2} \]

i.c.’s are \( u(x, 0) = u_0(x) \)
\[ \frac{\partial u}{\partial t}(x, 0) = \dot{u}_0(x) \]

b.c.’s are \( u(x, t) = \bar{u}(x, t) \) on \( \Gamma_1 \) (essential)
\[ \lambda \frac{\partial u}{\partial n}(x, t) = \bar{g}(x, t) \) on \( \Gamma_2 \) (natural)
Approach 1

- Use Galerkin in space, FD in time
- Develop the fundamental weak weighted residual form:

\[
\iint_{\Omega} \left( \lambda \nabla^2 u - \frac{\partial^2 u}{\partial t^2} \right) w_j d\Omega - \iint_{\Gamma_N} \left( \lambda \frac{\partial u}{\partial n} - \vec{g} \right) w_j d\Gamma = 0 \quad j = 1, N
\]

This generates a differentially time dependent set of simultaneous algebraic equations with the time dependence in the \( \alpha_i(t) \)'s

\[
\frac{\partial^2 \alpha}{\partial t^2} = \frac{A}{B} \frac{\partial^2 \alpha}{\partial t^2} + \frac{B}{\alpha} = P
\]

Now we use the FD method to resolve this differential time dependence leading to a system of entirely algebraic equations. Thus we discretize \( \frac{\partial^2 \alpha}{\partial t^2} \).

Approach 2

- FE for time and space:

\[
\int_0^{\Delta t} \left\{ \left( \iint_{\Omega} \left( \lambda \nabla^2 u - \frac{\partial^2 u}{\partial t^2} \right) w_j d\Omega - \iint_{\Gamma_N} \left( \lambda \frac{\partial u}{\partial n} - \vec{g} \right) w_j d\Gamma \right) \right\} dt = 0
\]