Lecture No. 17

Standard Galerkin Solutions to the Convection-Diffusion Equation

\[ \frac{\partial u}{\partial t} + V \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial u}{\partial x} \right) \]

- For the standard Galerkin method, the weighting function equals the interpolating functions.
- The standard Galerkin technique is also referred to as the Bubnov-Galerkin method.
- Typically, we use the FEM only to resolve the differential spatial dependence.
  This leads to:

\[ Mu_t + (A + B)u = P \]

- Now, we apply the FD method to resolve the remaining differential time dependence. Using a weighted implicit/explicit scheme we have:

\[ \{M + \Delta t \theta (A_{j+1} + B_{j+1})\}u_{j+1} = Mu_j + (\theta - 1)\Delta t (A_j + B_j)u_j + \Delta t \theta P_{j+1} + \Delta t (1 - \theta)P_j \]
Linear Finite Elements

Applying linear elements we will obtain the following general nodal equation:

\[
\frac{1}{6} \left\{ \frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta t} + 4 \frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{u_{i-1,j+1} - u_{i-1,j}}{\Delta t} \right\} \\
\frac{V}{2\Delta x} \left\{ \theta [u_{i+1,j+1} - u_{i-1,j+1}] + (1 - \theta) [u_{i+1,j} - u_{i-1,j}] \right\} \\
- \frac{D}{\Delta x^2} \left\{ \theta [u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}] + (1 - \theta) [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \right\} = 0
\]

Lumping, we have the following nodal equation:

\[
\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \frac{V}{2\Delta x} \left\{ \theta [u_{i+1,j+1} - u_{i-1,j+1}] + (1 - \theta) [u_{i+1,j} - u_{i-1,j}] \right\} \\
- \frac{D}{\Delta x^2} \left\{ \theta [u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}] + (1 - \theta) [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \right\} = 0
\]

This equation for the lumped linear FE formulation is identical to that obtained for the central difference FD formulation.
Comparisons between consistent finite element (FE) and lumped finite elements FE-L/FD solutions

A. Fully explicit $\theta = 0.0$

1. Pure Convection $P_e = \infty$
   
   i. Fig. L29.1a 1/2: Amplitude Portraits
      
      - Shows that there is no damping for either FE-L/FD or FE solutions for any $C_\#$ values.
      - Thus both methods are unconditionally unstable at $P_e = \infty$.
      
      Recall that stable solutions require $|\xi_n'\rangle \leq 1$.
      
      - FE seems to be more severely unstable (i.e. it will grow unstable more quickly).

   ii. Fig. L29.1b 1/2: Ratio Portraits
      
      - Shows that $|\xi_n'|/|\xi_n'\rangle > 1$ $\forall C_\#$ values. Thus the numerical solution is under-damped as compared to the analytical solution.
      
      - Recall that a perfect solution has a ratio = 1 and an overdamped solution has ratio $<< 1$. 
iii. Fig. L29.1b 1/2: Phase Portraits

- Shows that FE has superior phase propagation characteristics as compared to FE-L/FD. Therefore the numerical dispersion properties are better and the wiggles are less severe.
- Recall perfect solution has phase lag = 0.

iv. Fig. L29.2a 1/2: Computed Solution at $C_# = 0.1, P_e \infty$

- Both solutions show statically unstable behavior.
- Instability grows more quickly in the FE solution than FE-L/FD solution
- These features are predicted by the amplitude portrait.

v. Fig. L29.2b 1/2: Computed Solution at $C_# = 1.1, P_e \infty$

- Dynamically unstable behavior for both solutions
- Predicted by amplitude portrait

Note: Table L29.1: defines error measures given on all computed results
2. **Convection/Diffusion:** \( \textbf{P}_e = 2 \)

i. **Fig. L29.3a 1/2: Amplitude Portrait**

- Indicates that FE-L/FD is stable for \( \textbf{C}_# \leq 1.0 \) at \( \textbf{P}_e = 2 \)
- FE solution seems to be unstable for \( \textbf{C}_# \geq 0.5 \) (or lower ?) at \( \textbf{P}_e = 2 \). It is stable for \( \textbf{C}_# = 0.1 \)
- For stable regions the solutions will be damped

ii. **Fig. L29.3b 1/2: Ratio Portrait**

- Solutions are for the most part underdamped
- FE-L/FD and FE are very similar except at \( \textbf{C}_# = 0.1 \), the FE-L/FD underdamps whereas FE overdamps but only at high wave numbers.

iii. **Fig. L29.3c 1/2: Phase Portrait**

- Indicates that FE-L/FD has excellent phase propagation characteristics except at \( \textbf{C}_# = 0.1 \).
• FE also has good phase propagation characteristics except at $C_# = 0.5$ where there is a phase lead.

iv. Fig. L29.4a 1/2: Computed Solution at $P_e = 2$ and $C_# = 0.1$

• Both solution are good, FE is slightly superior over FE-L/FD.
• Slight overall phase lag for FE-L/FD and slight phase lead for FE as indicated by the phase portraits.
• Underdamping is not a problem as is indicated by Ratio portraits
• Any wiggles introduced will be damped out due to amplitude damping as indicated by amplitude portrait (physical damping present).

v. Fig. L29.4b 1/2: Computed Solution at $P_e = 2$ and $C_# = 1.1$

• Dynamically unstable
• FE much more so than FE-L/FD. This was indicated by amplitude portraits.

3. General Conclusions FE-L/FD and FE explicit
- Stability is a severe problem for both FE-L/FD and FE schemes when an explicit time discretization is used. Always unstable for convection dominant cases.
- FE has more restrictive ability characteristics than FE-L/FD. Also unstable behavior for FE is more severe (but this really doesn’t matter).
- Diffusion dominant low $C_#$ problems handled reasonably well (less restrictive $C_#$ conditions for FE-L/FD than FE).

B. **Crank-Nicolson** $\theta = 0.5$

1. **Pure Convection**: $P_e = \infty$ 

   i. **Fig. L29.5a 1/2 and L29.5b 1/2**: Amplitude Ratio Portraits

   - Shows no damping and perfect analytical damping for all $C_#$ values for both FE-L/FD and FE
   - Thus no stability constraints and no under or over damping.
   - Thus amplitude and ratio values are exact or perfect.

   ii. **Fig. L29.5c 1/2**: Phase Portraits
• Shows that FE has much better phase propagations characteristics than FE-L/FD. This is especially true of lower $C_\#$ values ($C_\# = 0.1, 0.5$) although it is in general the case.

iii. Fig. L29.6a 1/2: Computed Solution at $P_e = \infty$ and $C_\# = 0.1$

• FE solution has fewer and much smaller wiggles
• FE solution has a peak which is well propagated while FE-L/FD solution has severe peak phase lag.
• These differences result from the better phase behavior of the FE scheme.

iv. Fig. L29.6b 1/2: Computed Solution at $P_e = \infty$ and $C_\# = 1.1$

• FE solution is again better.
However difference is not as large as at $C_\# = 0.1$ since although phase behavior of FE is better it is not drastically better than FE-L/FD at high $C_\#$ values.

2. Convection/Diffusion: $P_e = 2$

i. Fig. L29.7a 1/2: Amplitude Portrait
• All cases damp for all wavelengths
• Thus stable and will help damn wiggles

ii. Fig. L29.7b 1/2: Ratio Portrait

• For FE-L/FD, all cases damp less than analytical solution. Thus there is no over-damping whatsoever.
• For FE low \( C_\# \) values have some overdamping but only at low wavelengths. High \( C_\# \) values behave like the FE-L/FD solution.

iii. Fig. L29.7c 1/2:

• FE has much superior phase properties compared to FE-L/FD.
• FE has slight phase lead behavior at higher \( C_\# \) values but only at very low wavelength ratios.

iv. Fig. L29.8

• FE is superior to FE-L/FD.
• FE-L/FD exhibits slight overall phase lag due to poorer phase behavior.
• Because of longer wavelengths involved in this problem, no substantial phase lag problems.

• Also any wiggles initially generated due to phase errors will be damped out due to physical damping present.

v. Fig. L29.8b 1/2: Computed Solutions at $P_e = 2$ and $C_# = 1.1$

• FE again superior to FE-L/FD.

• Both solutions have deteriorated as compared to low $C_#$ values, but only slightly so.

3. General Conclusions FE-L/FD and Fe with Crank-Nicolson

• Unconditionally stable for both

• Accuracy is much superior than FE than for FE-L/FD mainly due to much better phase behavior.

In highly physically damped cases this leads to better overall phase of the distribution (with no wiggles present in either solution).
C. **Fully Implicit** \( \theta = 1.0 \)

1. **Pure Convection**: \( P_e = \infty \)
   
i. Fig. L29.9a 1/2:
   
   - All wavelengths are damped for \( P_e = \infty \) and \( C_# = 0.1 \rightarrow 1.1 \).
   - Thus we can expect stable solutions which are damped for both FE-L/FD and FE.

   ii. Fig. L29.9b 1/2: Ratio Portrait
   
   - Numerical solutions are overdamped as compared to analytical solutions.
   - For \( C_# = 0.1 \) this overdamping is not severe and is slightly greater at small wavelengths in FE than for FE-L/FD.
   - For higher \( C_# \) values (0.5 \( \rightarrow \) 1.1), overdamping is severe for the entire wavelength range and is similar for both FE-L/FD and FE.
iii. Fig. L29.9c 1/2: Phase Portrait

- Shows that FE phase behavior is much better than FE-L/FD. This is especially true for lower $C_\#$ values (0.1 → 0.5).

iv. Fig. L29.10a 1/2: Computed Solution at $P_e = \infty$ and $C_\# = 0.1$

- FE-L/FD solution is very poor exhibiting wiggles which are not sufficiently damped.
- Overall phase is also very poor.
- These wiggles are due to FE-L/FD having poor phase behavior in important wavelength range while there is not sufficient damping to get rid of them.
- FE solution has no wiggles and overall phase is excellent.
- However the solution is overdamped. This is due to the fact that the phase propagation characteristics are very good and we have sufficient damping for the very short wavelengths. However we’ve lost parts of our solution through this combined dispersion and damping process and this accounts for the overdamped solution.
v. Fig. L29.10b 1/2: Computed Solution at $P_e = \infty$ and $C_# = 1.1$

- Both FE-L/FD and Fe solutions are severely overdamped. This feature is predicted by ratio portraits.

2. Convection/Diffusion $P_e = 2$

i. Fig. L29.11a 1/2: Amplitude Portrait

- Shows that both FE-L/FD and Fe always damp and thus have stable behavior for all $C_#$ values considered.

ii. Fig. L29.11b 1/2: Ratio Portraits

- Shows that for all higher $C_#$ values (0.5 – 1.1) there is severe overdamping in the high wavelength range.
- At $C_# = 0.1$, behavior is much better and only the FE shows damping at very low wavelength values.
iii. Fig. L29.11c 1/2: Phase Portrait
   - FE has superior phase behavior, especially at lower $C_\#$ values.

iv. Fig. L29.12a 1/2: Computed Solution at $P_e = \infty$ and $C_\# = 0.1$
   - Both FE-L/FD and FE solutions are reasonable
   - FE solution has slightly better overall phase.

v. Fig. L29.12b 1/2: Computed Solution at $P_e = \infty$ and $C_\# = 1.1$
   - Both FE-L/FD and FE are severely overdamped. This is predicted by ratio portrait.

3. General Conclusions FE-L/FD and FE fully implicit
   - FE is somewhat superior in that it is slightly more accurate and eliminates wiggles better.
   - In general, implicit methods for both FE-L/FD and FE often lead to severely overdamped results.
D. General Conclusions – sections A, B, C

- Best time integration scheme is Crank-Nicolson. Explicit methods were generally unstable for convection dominated flows while implicit methods have severe overdamping problems. Recall that C-N is $O(\Delta t)^2$ accurate whereas explicit and implicit methods were only $O(\Delta t)$ accurate.

- FE is much better than FE-L/FD for C-N in terms of overall quality of results. Quality is excellent for low $P_e$ and good for high $P_e$ (some wiggles remained).

- If you’re applying consistent FE is it always computationally worthwhile to use C-N since you’re backsubstituting into matrices anyway (although matrices may have to be reset).

- Always exercise extreme caution with any numerical solution.
Table L29.1

**Error E1:** Integral measure of the overall error of the numerical solution.

\[
E1 = \frac{1}{m(t)} \left[ \int_{0}^{L} (\phi_{\text{num}}(x, t) - \phi_{\text{ex}}(x, t))^2 dx \right]^{1/2}
\]

Value for Exact Solution = 0.0

**Error E2:** Discrete measure of the overall error of the numerical solution.

\[
E2 = \frac{1}{m(t)} \left[ \sum_{i} (\phi_{\text{num}}^i(t) - \phi_{\text{ex}}(x_i, t))^2 \right]^{1/2}
\]

Value for Exact Solution = 0.0

**Error E3:** Point measure of the artificial damping of the numerical solution (peak depression).

\[
E3 = \left\| \frac{\phi_{\text{ex}}(x, t) - \phi_{\text{num}}(x, t)}{\phi_{\text{ex}}_{\text{max}}(t)} \right\|
\]

Value for Exact Solution = 0.0
Error E4: Point measure of the maximum spurious oscillation in the numerical solution.

\[ E_4 = \left| \frac{\phi_{\text{num, neg}}(t)}{\phi_{\text{max}}(t)} \right| \]

Value for Exact Solution = 0.0

Error E5: Point measure of the phase shift introduced in the numerical solution.

\[ E_5 = \frac{x_{\text{max}}^{\text{ex}}(t) - x_{\text{max}}^{\text{num}}(t)}{x_{\text{max}}^{\text{ex}}(t)} \]

Value for Exact Solution = 0.0

Error E6: Integral measure of mass preservation.

\[ E_6 = \frac{1}{m(t)} \int_0^L \phi^{\text{num}}(x, t) dx \]

Value for Exact Solution = 1.0
Fig. L29.1a1

Finite Element Methods

F.E. Lumped Fully Explicit ($\theta = 0.0$) $\beta = \infty$

Modulus of Amplification Factor

Wavelength/DELX

- Line 1
- Line 2
- Line 3
- Line 4

Different curves for $C = 0.1$, $C = 0.5$, $C = 1.0$, $C = 1.1$
Fig. L29.1a2

F.E. CONSISTENT FULLY EXPLICIT ($\theta = 0.0$) $P = \omega$

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX

$\Delta = 0.1$

$\Delta = 0.5$

$\Delta = 1.0$

$\Delta = 1.1$

Lines:
1
2
3
4
Fig. L29.1b1

F.E. LUMPED FULLY EXPPLICIT (θ=0.0) |P=∞

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX

\( \theta = 1.1 \)

\( \theta = 1.0 \)

\( \theta = 0.5 \)

\( \theta = 0.1 \)
Fig. L29.1b2

F.E. CONSISTENT FULLY EXPLICIT ($\Theta=0.0$) $P=\infty$

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/$\Delta$lx

- 1
- 2
- 3
- 4

$\Theta=0.1$

$\Theta=0.5$

$\Theta=1.0$

$\Theta=1.1$
Fig. L29.1c1

F.E. LUMPED  FULLY EXPLICIT ($\theta = 0.0$)  $\beta = \omega$

Phase lag vs. Wavelength/DELX for different values of $\zeta$:
Fig. L29.1c2

F.E. CONSISTENT FULLY EXPLICIT (θ=0.0) \( P = \infty \)

![Graph of phase lag vs. wavelength/\( \Delta x \)]
PROBLEM SET #1A WITH LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 0.10  PECELT NO: INFINITY
DEGREE OF BASIS POLYNOMIAL=1

\[
\begin{align*}
\text{ALPHA} &= 0.00 & \text{ERROR 1} &= 0.039162 \\
\text{BETA} &= 0.00 & \text{ERROR 2} &= 0.002948 \\
\text{ALPHAM} &= 0.00 & \text{ERROR 3} &= 0.191188 \\
\text{ALPHAC} &= 0.00 & \text{ERROR 4} &= 0.645687 \\
\text{BETAM} &= 0.00 & \text{ERROR 5} &= 0.058824 \\
\text{BETAC} &= 0.00 & \text{ERROR 6} &= 0.995453 \\
\text{ERROR 7} &= 0.000678 & \text{ERROR 8} &= 1.068876
\end{align*}
\]
FULLY EXPlicit TIME INTEGRATION
COURANT NO: 0.10  .  PECELT NO: INFINITY
DEGREE OF BASIS POLYNOMIAL: 1

ALPHA = 0.00
BETA = 0.00
ALPHAM = 0.00
ALPHA = 0.00
ALPHAC = 0.00
BETAM = 0.00
BETA = 0.00

ERROR 1 = 0.051337
ERROR 2 = 0.004411
ERROR 3 = 0.029243
ERROR 4 = 1.15982
ERROR 5 = 0.029412
ERROR 6 = 1.005484
ERROR 7 = -0.00109
ERROR 8 = 0.546769
Fig. L29.2b1

PROBLEM SET #1A WITH LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 1.10
PECLET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL: 1

ALPHA = 0.00
BETA = 0.00
ALPHA = 0.00
ALPHA = 0.00
BETAM = 0.00
BETAC = 0.00

ERROR 1 = 0.02951
ERROR 2 = 0.002334
ERROR 3 = 0.559632
ERROR 4 = 0.698777
ERROR 5 = 0.097744
ERROR 6 = 1.000001
ERROR 7 = 0.000000
ERROR 8 = -0.987714

CONCENTRATION

X-COORDINATE
Fig. L29.2b2

PROBLEM SET #1A NO LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 1.10  REYNOLDS NO: INFINITY
DEGREE OF BASIS POLYNOMIAL: 1

ALPHA = 0.00
BETA = 0.00
ALPHA = 0.00
ALPHA = 0.00
ALPHA = 0.00
BETA = 0.00
BETA = 0.00

ERROR 1 = 0.039210
ERROR 2 = 0.003157
ERROR 3 = 0.04711
ERROR 4 = 0.928769
ERROR 5 = 0.097744
ERROR 6 = 0.000003
ERROR 7 = -0.000001
ERROR 8 = -0.987956

CONCENTRATION

X-COORDINATE
Fig. L29.3a1

F.E. LUMPED FULLY EXPLICIT (θ=0.0) \( \Pi=2 \)

![Graph showing the modulus of amplification factor vs. wavelength/delx for different values of \( \lambda \).](image-url)
F.E. CONSISTENT FULLY EXPLICIT (Θ=0.0)  \( \nu=2 \)
Fig. L29.3b1

F.E. LUMPED FULLY EXPLICIT ($\theta=0.0$) $P=2$

![Graph showing ratio of amplification factors versus wavelength/DELX with different lines and labels for specific values of $\xi$.](image-url)
Fig. L29.3b2

F.E. CONSISTENT FULLY EXPLICIT (θ=0.0) P=2

- - - - 1
- - - - 2
- - - - 3
- - - - 4

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX

λ=0.1
λ=0.5
λ=1.0
λ=1.1
Fig. L29.3c1

F.E. LUMPED FULLY EXPLICIT (θ = 0.0) P = 2

PHASE LAG

WAVELENGTH/DELEX

1
2
3
4
Fig. L29.3c2

F.E. CONSISTENT FULLY EXPLICIT ($\theta=0.0$) $IP=2$

![Graph showing phase lag vs wavelength/DELX with different curves for $c=0.1$, $c=0.5$, $c=1.0$, and $c=1.1$. Each curve is labeled with its corresponding $c$ value.](image-url)
Fig. L29.4a1

PROBLEM SET #1A WITH LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 0.10  PECELT NO: 2.000
DEGREE OF BASIS POLYNOMIAL = 1

\begin{align*}
\text{ALPHA} &= 0.00 \\
\text{BETA} &= 0.00 \\
\text{ALPHA}_M &= 0.00 \\
\text{ALPHA}_C &= 0.00 \\
\text{BETA}_M &= 0.00 \\
\text{BETA}_C &= 0.00 \\
\end{align*}

\begin{align*}
\text{ERROR 1} &= 0.000000 \\
\text{ERROR 2} &= 0.000000 \\
\text{ERROR 3} &= 0.000000 \\
\text{ERROR 4} &= 0.000000 \\
\text{ERROR 5} &= 0.000000 \\
\text{ERROR 6} &= 1.000000 \\
\text{ERROR 7} &= 0.000000 \\
\text{ERROR 8} &= 0.913240 \\
\end{align*}
Fig. L29.4a2

PROBLEM SET #1A NO LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 0.10, PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL = 1
ALPHA = 0.00
BETA = 0.00
ALPHA_H = 0.00
ALPHA_C = 0.00
BETA_H = 0.00
BETA_C = 0.00
ERROR 1 = 0.000624
ERROR 2 = 0.000048
ERROR 3 = 0.002757
ERROR 4 = 0.000001
ERROR 5 = 0.000000
ERROR 6 = 1.000000
ERROR 7 = 0.000000
ERROR 8 = 0.913240

CONCENTRATION

X-COORDINATE

0 1000 2000 3000 4000 5000 6000 7000 8000 9000
Fig. L29.4b1

FULLY EXPLICIT TIME INTEGRATION
COURANT NO: 1.10, PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL = 1

USING:

\[
\begin{align*}
\text{ALPHA} &= 0.00 \\
\text{BETA} &= 0.00 \\
\text{ALPHAM} &= 0.00 \\
\text{ALPHAC} &= 0.00 \\
\text{BETAM} &= 0.00 \\
\text{BETAC} &= 0.00 \\
\end{align*}
\]

ERROR:
1 = 0.47913
2 = 0.003774
3 = 6.435181
4 = 4.007101
5 = 0.033233
6 = 1.000000
7 = 0.000000
8 = -0.016142

PROBLEM SET #1A WITH LUMPING

\[
\begin{align*}
n & = 1, \ldots, N
\end{align*}
\]
Fig. L29.4b2

PROBLEM SET #1A NO LUMPING

FULLY EXPLICIT TIME INTEGRATION
COURANT NO. 1.10, PECELT NO. 2.000
DEGREE OF BASIS POLYNOMIAL 1

ALPHA = 0.00
BETA = 0.00
ALPHA_H = 0.00
ALPHA_C = 0.00
BETA_H = 0.00
BETA_C = 0.00

ERROR 1 = 0.583845
ERROR 2 = 0.064364
ERROR 3 =
ERROR 4 =
ERROR 5 = 0.154589
ERROR 6 = 0.000003
ERROR 7 = -0.007637
ERROR 8 = 0.150491

CONCENTRATION

X-COORDINATE

0 1000 2000 3000 4000 5000 6000 7000 8000 9000
Fig. L29.5a1

F.E. LUMPED \( C-N \) (\( \theta = 0.5 \)) \( |P| = \infty \)

<table>
<thead>
<tr>
<th></th>
<th>MODULUS OF AMPLIFICATION FACTOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

\( \alpha = 0.1, 0.5, 1.0, 1.1 \)

WAVELENGTH/DELX
Fig. L29.5a2

F.E. consistent, C-N (θ=0.5) \( P=\infty \)

- - - - 1
- - - 2
- - - - 3
--- 4

\[ \lambda = 0.1, 0.5, 1.0, 1.1 \]

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX
Fig. L29.5b1

F.E. LUMPED C-N (θ = 0.5) \( P = \infty \)

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX

c = 0.1, 0.5, 1.0, 1.1
Fig. L29.5b2

F.E. CONSISTENT, C-N (θ=0.5) \( IP=\infty \)

- - - 1
- - - 2
- - - 3
- - - 4

\( \lambda = 0.1, 0.5, 1.0, 1.1 \)

Ratio of Amplification Factors

Wavelength/Δlx
Fig. L29.5c1

F.E. LUMPED  C-N (θ = 0.5)  IP = ∞
Fig. L29.5c2

F.E. CONSISTENT, C-N ($\theta=0.5$), $P=\infty$

![Graph showing phase lag vs. wavelength/DELX](image-url)
PROBLEM SET #1A  WITH LUNMIE

CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 0.10  ,  PECELET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL=1

ALPHA=0.00
BETA=0.00
ALPHA_H=0.00
ALPHA_C=0.00
BETA_H=0.00
BETA_C=0.00

ERROR 1=0.025127
ERROR 2=0.001833
ERROR 3=0.332301
ERROR 4=0.350483
ERROR 5=0.058824
ERROR 6=0.998291
ERROR 7=0.000381
ERROR 8=1.614065
FIG. L29.6a2

PROBLEM SET #1A NO LUMPING

CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 0.10 , PECELET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

\begin{align*}
\text{ALPHA} &= 0.00 & \text{ERROR} 1 &= 0.005167 \\
\text{BETA} &= 0.00 & \text{ERROR} 2 &= 0.000387 \\
\text{ALPHAM} &= 0.00 & \text{ERROR} 3 &= 0.0814728 \\
\text{ALPHAC} &= 0.00 & \text{ERROR} 4 &= 0.097398 \\
\text{BETAM} &= 0.00 & \text{ERROR} 5 &= 0.00000 \\
\text{BETAC} &= 0.00 & \text{ERROR} 6 &= 1.000055 \\
\text{ERROR} 7 &= 0.000098 \\
\text{ERROR} 8 &= 1.094053
\end{align*}

\[ \text{CONCENTRATION} \]

\[ \text{X-COORDINATE} \]
PROBLEM SET #1A WITH LUMPING

CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 1.10 , PECLET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

ALPHA = 0.00
BETA = 0.00
ALPHAM = 0.00
ALPHAC = 0.00
BETAM = 0.00
BETAC = 0.00

ERROR 1 = 0.027912
ERROR 2 = 0.002028
ERROR 3 = 0.384000
ERROR 4 = 0.385270
ERROR 5 = 0.063444
ERROR 6 = 0.998236
ERROR 7 = 0.000141
ERROR 8 = -0.998961
Fig. L29.6b2

PROBLEM SET #1A NO LUMPING

CRANK-NICOLSON TIME INTEGRATION
COURANT NO.: 1.10 , PECLET NO.: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALPHA = 0.00</td>
<td>ERROR 1 = 0.020304</td>
</tr>
<tr>
<td>BETAN = 0.00</td>
<td>ERROR 2 = 0.001483</td>
</tr>
<tr>
<td>ALPHAN = 0.00</td>
<td>ERROR 3 = 0.251336</td>
</tr>
<tr>
<td>ALPHAC = 0.00</td>
<td>ERROR 4 = 0.336473</td>
</tr>
<tr>
<td>BETAN = 0.00</td>
<td>ERROR 5 = 0.033233</td>
</tr>
<tr>
<td>BETAN = 0.00</td>
<td>ERROR 6 = 0.999673</td>
</tr>
<tr>
<td>BETAN = 0.00</td>
<td>ERROR 7 = 0.000118</td>
</tr>
<tr>
<td>BETAN = 0.00</td>
<td>ERROR 8 = 1.126360</td>
</tr>
</tbody>
</table>

CONCENTRATION

X-COORDINATE
Fig. L29.7a1

F.E. LUMPED C-N (θ=0.5) \( I_P = 2 \)

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX

\( c = 0.1 \)
\( c = 0.5 \)
\( c = 1.0 \)
\( c = 1.1 \)

1
2
3
4
Fig. L29.7a2

F.E. CONSISTENT C-N (θ = 0.5) \( IP = 2 \)

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX
Fig. L29.7b1

Finite Element Methods - Lecture 17

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX

FE. LUMPED \( C = \theta = 0.5 \) \( \rho = 2 \)
Fig. L29.7b2

F.E. consistent, C-N (θ=0.5), Ip=2

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX
Fig. L29.7c1

Finite Element Methods - Lecture 17

F.E. LUMPED C-N (θ = 0.5) \( \beta = 2 \)

\[
\begin{array}{c}
\text{PHASE LAG} \\
\hline
-7.5 & -5.0 & -2.5 & 0.0 & 2.5 & 5.0 & 7.5 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{WAVELENGTH}/\text{DELX} \\
\hline
10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \\
\hline
\end{array}
\]

\( C = 0.1 \)
\( C = 0.5 \)
\( C = 1.1 \)
Fig. L29.7c2

F.E. CONSISTENT, C - N (θ = 0.5), kP = 2

- - - 1
- - - - 2
--- - 3
--- - - 4

<table>
<thead>
<tr>
<th>PHASE LAG</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-4</td>
</tr>
<tr>
<td>-6</td>
</tr>
<tr>
<td>-8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>WAVELENGTH/DELX</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>70</td>
</tr>
<tr>
<td>80</td>
</tr>
<tr>
<td>90</td>
</tr>
<tr>
<td>100</td>
</tr>
</tbody>
</table>
CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 0.10  PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=0.00
BETA=0.00
ALPHA=0.00
ALPHA=0.00
ALPHA=0.00
BETA=0.00
BETA=0.00
BETA=0.00

PROBLEM SET #1A WITH LUMPING

CONCENTRATION
0.00 0.20 0.40 0.60 0.80 1.00 1.20

X-COORDINATE
0 1000 2000 3000 4000 5000 6000 7000 8000 9000

ERROR 1=0.000071
ERROR 2=0.000050
ERROR 3=0.00248
ERROR 4=0.000000
ERROR 5=0.000000
ERROR 6=1.000000
ERROR 7=0.000000
ERROR 8=1.006465
Fig. L29.8a2

PROBLEM SET #1A NO LUMPING

CRANK-NICOLSON TIME INTEGRATION
COURANT NO. 0.10, PECLET NO. 2.000
DEGREE OF BASIS POLYNOMIAL = 1

<table>
<thead>
<tr>
<th>ALPHA</th>
<th>ERROR 1</th>
<th>0.000106</th>
</tr>
</thead>
<tbody>
<tr>
<td>BETA</td>
<td>ERROR 2</td>
<td>0.000004</td>
</tr>
<tr>
<td>ALPHAM</td>
<td>ERROR 3</td>
<td>0.004311</td>
</tr>
<tr>
<td>ALPHAC</td>
<td>ERROR 4</td>
<td>0.000000</td>
</tr>
<tr>
<td>BETAM</td>
<td>ERROR 5</td>
<td>0.000000</td>
</tr>
<tr>
<td>BETAC</td>
<td>ERROR 6</td>
<td>1.000000</td>
</tr>
<tr>
<td></td>
<td>ERROR 7</td>
<td>0.000000</td>
</tr>
<tr>
<td></td>
<td>ERROR 8</td>
<td>1.006472</td>
</tr>
</tbody>
</table>

CONCENTRATION

X-COORDINATE
Fig. L29.8b1

CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 1.10, PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL = 1

ALPHA = 0.00
BETA  = 0.00
ALPHA = 0.00
BETA  = 0.00
ALPHA = 0.00
BETA  = 0.00
ALPHA = 0.00
BETA  = 0.00

ERROR 1 = 0.001137
ERROR 2 = 0.000082
ERROR 3 = 0.011535
ERROR 4 = 0.000000
ERROR 5 = 0.033233
ERROR 6 = 0.999999
ERROR 7 = 0.000002
ERROR 8 = 1.006677

PROBLEM SET #1A WITH LUMPING
Fig. L29.8b2

PROBLEM SET #1A NO LUMPING

CRANK-NICOLSON TIME INTEGRATION
COURANT NO: 1.10 , PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=0.00                ERROR 1=0.000414
BETA=0.00                  ERROR 2=0.000030
ALPHAH=0.00                ERROR 3=0.011032
ALPHAC=0.00                ERROR 4=0.000000
BETAH=0.00                 ERROR 5=0.003021
BETAC=0.00                 ERROR 6=1.000000
ERROR 7=0.000000
ERROR 8=1.006702

CONCENTRATION

0.00  0.20  0.40  0.60  0.80  1.00  1.20

0  1000  2000  3000  4000  5000  6000  7000  8000  9000
X-COORDINATE
Fig. L29.9a1

F.E. LUMPED FULLY IMPLICIT ($\theta = 1.0$) $\beta = \infty$

![Graph showing the modulus of amplification factor as a function of wavelength/DELX for different values of $\zeta$.](image-url)
Fig. L29.9a2

F.E. CONSISTENT FULLY IMPLICIT \((\theta=1.0)\) \(P=\infty\)

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX
Fig. L29.9b1

F.E. LUMPED   FULLY IMPLICIT ($\theta = 1.0$)   $\lambda = \infty$

RATIO OF AMPLIFICATION FACTORS

WAVELENGTH/DELX

$\zeta = 0.1$
$\zeta = 0.5$
$\zeta = 1.0$
$\zeta = 1.1$
Fig. L29.9b2

F.E. CONSISTENT FULLY IMPLICIT (θ = 1.0) \( |P| = \infty \)

- - - - 1
- - - - 2
- - - - 3
- - - - 4

RATIO OF AMPLIFICATION FACTORS

\( c = 0.1 \)
\( c = 0.5 \)
\( c = 1.0 \)
\( c = 1.1 \)

WAVELENGTH/DELX
Fig. L29.9c1

F.E. LUMPED FULLY IMPLICIT (θ=10) IP=∞
Fig. L29.9c2

F.E. CONSISTENT FULLY IMPLICIT (θ = 1.0) \( \lambda_P = \omega \)

![Graph showing phase lag vs. wavelength/\( \Delta x \)]

- \( \zeta = 0.1 \)
- \( \zeta = 0.5 \)
- \( \zeta = 1.0 \)

Axis labels:
- Y-axis: PHASE LAG
- X-axis: WAVELENGTH/\( \Delta x \)
FULLY IMPLICIT TIME INTEGRATION
COURANT NO:0.10  ,  PECLET NO:INFINITY
DEGREE OF BASIS POLYNOMIAL=1

ALPHA=0.00  ERROR 1=0.020703
BETA=0.00   ERROR 2=0.001480
ALPHAM=0.00 ERROR 3=0.441102
ALPHAC=0.00 ERROR 4=0.100759
BETAM=0.00  ERROR 5=0.058824
BETAC=0.00  ERROR 6=0.999063
ERROR 7=0.000278
ERROR 8=2.571612
Fig. L29.10a2

PROBLEM SET #1A NO LUMPING

FULLY IMPLICIT TIME INTEGRATION
COURANT NO: 0.10, PECLET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

\[
\begin{array}{cccc}
\text{ALPHA} & = & 0.00 & \text{ERROR 1} = 0.011352 \\
\text{BETA} & = & 0.00 & \text{ERROR 2} = 0.000775 \\
\text{ALPHA} & = & 0.00 & \text{ERROR 3} = 0.356333 \\
\text{ALPHA} & = & 0.00 & \text{ERROR 4} = 0.001616 \\
\text{BETA} & = & 0.00 & \text{ERROR 5} = 0.000000 \\
\text{BETA} & = & 0.00 & \text{ERROR 6} = 0.999883 \\
\text{BETA} & = & 0.00 & \text{ERROR 7} = 0.000106 \\
\text{BETA} & = & 0.00 & \text{ERROR 8} = 2.462141 \\
\end{array}
\]
Fig. L29.10b1

PROBLEM SET #1A WITH LUMPING

FULLY IMPLICIT TIME INTEGRATION
COURANT NO: 1.10, PECLET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

ALPHA = 0.00
BETA = 0.00
ALPHAM = 0.00
ALPHAC = 0.00
BETAM = 0.00
BETAC = 0.00

ERROR 1 = 0.024756
ERROR 2 = 0.001749
ERROR 3 = 0.71442
ERROR 4 = 0.000672
ERROR 5 = 0.033233
ERROR 6 = 0.999630
ERROR 7 = 0.000233
ERROR 8 = *********

CONCENTRATION

α-COORDINATE

0 1000 2000 3000 4000 5000 6000 7000 8000 9000

-0.20 0.00 0.20 0.40 0.60 0.80 1.00 1.20
Fig. L29.10b2

PROBLEM SET #1A NO LUMPING

FULLY IMPLICIT TIME INTEGRATION
COURANT NO: 1.0, PECLET NO: INFINITY
DEGREE OF BASIS POLYNOMIAL = 1

\[
\begin{align*}
\text{ALPHA} &= 0.00 \\
\text{BETA} &= 0.00 \\
\text{ALPHAH} &= 0.00 \\
\text{ALPHAC} &= 0.00 \\
\text{BETAH} &= 0.00 \\
\text{BETAC} &= 0.00 \\
\text{ERROR 1} &= 0.024537 \\
\text{ERROR 2} &= 0.001733 \\
\text{ERROR 3} &= 0.741967 \\
\text{ERROR 4} &= 0.000018 \\
\text{ERROR 5} &= 0.033233 \\
\text{ERROR 6} &= 0.999974 \\
\text{ERROR 7} &= 0.000113 \\
\text{ERROR 8} &= \text{*****} \\
\end{align*}
\]
Fig. L29.11a1

F.E. LUMPED FULLY IMPPLICIT (θ = 1.0) IP = 2

MODULUS OF AMPLIFICATION FACTOR

WAVELENGTH/DELX

- 1
- 2
- 3
- 4
Fig. L29.11a2

F.E. CONSISTENT FULLY IMPLICIT \( (\theta = 1.0) \) \( \theta = 2 \)

---

**Graph Details**

- **Title:** F.E. CONSISTENT FULLY IMPLICIT \( (\theta = 1.0) \) \( \theta = 2 \)
- **Axes:**
  - Y-axis: MODULUS OF AMPLIFICATION FACTOR
  - X-axis: WAVELENGTH/DELX
- **Curves:**
  - \( \theta = 0.1 \)
  - \( \theta = 0.5 \)
  - \( \theta = 1.1 \)
  - \( \theta = 1.0 \)
- **Lines:**
  - Solid line for \( \theta = 0.1 \)
  - Dashed line for \( \theta = 0.5 \)
  - Dotted line for \( \theta = 1.1 \)
  - Dashed-dotted line for \( \theta = 1.0 \)
Fig. L29.11b1

F.E. LUMPED FULLY IMPLICIT (θ=1.0) \( \bar{P}=2 \)
Fig. L29.11b2

F.E. CONSISTENT FULLY IMPLICIT ($\theta = 1.0$) $IP = 2$
Fig. L29.11c1

F.E. LUMPED FULLY IMPLICIT (θ=1.0)  \( p = 2 \)

Phase lag

Wavelength/DELX
Fig. L29.11c2

F.E. CONSISTENT FULLY IMPLICIT ($\theta = 1.0$)  

\[ \text{PHASE LAG} \]

\[ \text{WAVELENGTH/DELX} \]
FULLY IMPLICIT TIME INTEGRATION
COURANT NO: 0.10, PECELT NO: 2.000
DEGREE OF BASIS POLYNOMIAL = 1

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>VALUE</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALPHA</td>
<td>0.00</td>
<td>0.001037</td>
</tr>
<tr>
<td>BETA</td>
<td>0.00</td>
<td>0.000072</td>
</tr>
<tr>
<td>ALPHAM</td>
<td>0.00</td>
<td>0.043911</td>
</tr>
<tr>
<td>ALPHAC</td>
<td>0.00</td>
<td>0.000000</td>
</tr>
<tr>
<td>BETAH</td>
<td>0.00</td>
<td>0.029412</td>
</tr>
<tr>
<td>BETAC</td>
<td>0.00</td>
<td>0.999999</td>
</tr>
<tr>
<td>ERROR</td>
<td>1-8</td>
<td></td>
</tr>
</tbody>
</table>

CONCENTRATION
0.00
0.20
0.40
0.60
0.80
1.00
1.20

X-COORDINATE
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

PROBLEM SET #1a WITH LUMPING
Problem Set #1A No Lumping

Fully Implicit Time Integration
Courant No.: 0.10, Peclet No.: 2.000
Degree of Basis Polynomial = 1

\begin{align*}
\text{Alpha} &= 0.00 \\
\text{Beta} &= 0.00 \\
\text{Alphac} &= 0.00 \\
\text{Betaac} &= 0.00 \\
\text{errors} &= \begin{align*}
\text{Error 1} &= 0.000734 \\
\text{Error 2} &= 0.000949 \\
\text{Error 3} &= 0.046165 \\
\text{Error 4} &= 0.000000 \\
\text{Error 5} &= 0.000000 \\
\text{Error 6} &= 1.000000 \\
\text{Error 7} &= 0.000000 \\
\text{Error 8} &= 1.099702
\end{align*}
\end{align*}
Fig. L29.12b1
FULLY IMPLICIT TIME INTEGRATION
COURANT NO: 1.10  ,  PECELT NO: 2.000
DEGREE OF BASIS POLYNOMIAL: 1

ALPHA = 0.00
BETA = 0.00
ALPHAM = 0.00
ALPHAC = 0.00
BETAM = 0.00
BETAC = 0.00

ERROR 1 = 0.004763
ERROR 2 = 0.000336
ERROR 3 = 0.285068
ERROR 4 = 0.000000
ERROR 5 = 0.033233
ERROR 6 = 0.999803
ERROR 7 = 0.000566
ERROR 8 = 2.620769

CONCENTRATION

0.00  0.20  0.40  0.60  0.80  1.00  1.20
0   1000  2000  3000  4000  5000  6000  7000  8000  9000

X-COORDINATE

Fig. L29.12b2