Lecture No. 3

The difference formulas used to approximate the various derivatives in a p.d.e. should all have ***equivalent order error***.

Order of error:

$$E \propto h^p \frac{d^q f}{dx^q}$$

Richardson Interpolation (Extrapolation)

With FD's, we find $\left.\frac{d^n f}{dx^n}\right|_{x = x_o}$ using an approximation.

The exact value equals the approximate value plus a given order error:

$$y_e = y_h + ch^p + dh^{p+1} + eh^{p+2} + \ldots.$$

where

y_e = the exact solution

y_h = the approximate value corresponding to spacing h

c, d, e = coefficients which depend on the derivative in the interval

Example Application of Richardson Interpolation

Let's assume that the FD operator is of second order accuracy, $p = 2$

- Now compute two solutions with different node spacings h_1 and h_2;

$$y_e = y_{h_1} + ch_1^2$$

$$y_e = y_{h_2} + ch_2^2$$

- c is approximately identical for both spacings.
- Note that the higher order error terms which we are not considering still exist.

- Solve for c using first equation:

$$c = \frac{1}{h_1^2}(y_e - y_{h_1})$$

- Now we have solved for the leading order error term for the given interval. By substituting into the second equation:

$$y_e = y_{h_2} + \left(\frac{1}{h_1^2}(y_e - y_{h_1})\right)h_2^2$$

$$\Rightarrow$$

$$\left(1 - \frac{h_2^2}{h_1^2}\right)y_e = y_{h_2} - \frac{h_2^2}{h_1^2}y_{h_1}$$

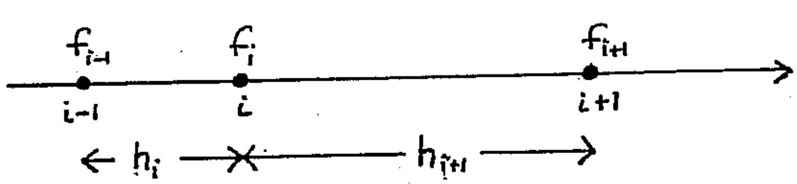
$$\Rightarrow$$

$$y_e = \frac{y_{h_1}}{(1 - h_1^2/h_2^2)} + \frac{y_{h_2}}{(1 - h_2^2/h_1^2)}$$

- This y_e is a better solution than y_{h_1} and y_{h_2}. Therefore, using two crude h values and using the <u>same</u> FD formula to obtain 2 estimates for the derivative, we get a much improved value which would otherwise require a much smaller node spacing than either h_1 or h_2.

<u>Unequal Spacing of Nodes</u>

Spacing between points need not be constant.



applying TS:

$$f_{i+1} = f_i + h_{i+1} f_i^{(1)} + \frac{1}{2} h_{i+1}^2 f_i^{(2)} + \frac{1}{6} h_{i+1}^3 f_i^{(3)} + \dots$$

$$f_{i-1} = f_i - h_i f_i^{(1)} + \frac{1}{2} h_i^2 f_i^{(2)} - \frac{1}{6} h_i^3 f_i^{(3)} + \dots$$

First Derivative Approximation (central) Using Unequal Nodal Spacing

Consider:

$$f_{i+1} - f_{i-1} = h_{i+1} f_i^{(1)} + h_i f_i^{(1)} + \frac{1}{2} (h_{i+1}^2 - h_i^2) f_1^{(2)} + \frac{1}{6} (h_{i+1}^3 + h_i^3) f_i^{(3)}$$

$$\Rightarrow$$

$$f_i^{(1)} = \frac{f_{i+1} - f_{i-1}}{(h_{i+1} + h_i)} - \frac{1}{2} (h_{i+1} - h_i) f_i^{(2)} - \frac{1}{6} \frac{(h_{i+1}^3 + h_i^3)}{(h_{i+1} + h_i)} f_i^{(3)} + \dots$$

$$\Rightarrow$$

$$f_i^{(1)} = \frac{f_{i+1} - f_{i-1}}{h_{i+1} + h_i} + E(\Delta h)$$

- Note that the leading truncation term is $E(\Delta h)$.
- Therefore the error order has increased. The error increases with larger variation in node spacing.
- If $h_{i+1} = h_i$, the error reduces to that of the 2nd order central difference formula.
- The order of the polynomial for which the expression is exact has also decreased (to linear from quadratic for constant spacing).
- Variable spacing can be very useful to control the level of accuracy of the solu-

tion since the truncation error terms equal the product of an $E(\Delta h)$ or $E(h^n)$ term and a spatial derivative of the solution. Thus when the solution varies rapidly in space, the nodal spacing can be reduced in order to control the magnitude of the various truncation terms and thus the overall accuracy of the solution.

2nd Derivative Approximation (central) with Variable Nodal Spacing

Add f_{i+1} and $\frac{h_{i+1}}{h_i} f_{i-1}$

$$f_{i+1} + \frac{h_{i+1}}{h_i} f_{i-1} = f_i\left[1 + \frac{h_{i+1}}{h_i}\right] + \left[h_{i+1} - \frac{h_{i+1}h_i}{h_i}\right] f_i^{(1)}$$

$$+ \frac{1}{2}\left[h_{i+1}^2 + \frac{h_{i+1}}{h_i} h_i^2\right] f_i^{(2)} + \frac{1}{6}\left[h_{i+1}^3 - \frac{h_{i+1}}{h_i} h_i^3\right] f_i^{(3)}$$

$$\Rightarrow$$

$$f_i^{(2)} = \frac{f_{i+1} - \left(1 + \frac{h_{i+1}}{h_i}\right) f_i + \frac{h_{i+1}}{h_i} f_{i-1}}{\frac{1}{2}(h_{i+1}^2 + h_{i+1}h_i)} - \frac{1}{6}\frac{(h_{i+1}^3 - h_{i+1}h_i^2)}{\frac{1}{2}(h_{i+1}^2 + h_{i+1}h_i)} f_i^{(3)}$$

$$\Rightarrow$$

$$f_i^{(2)} = \frac{f_{i+1} - \left(1 + \frac{h_{i+1}}{h_i}\right) f_i + \frac{h_{i+1}}{h_i} f_{i-1}}{\frac{1}{2}(h_{i+1}^2 + h_{i+1}h_i)} - \frac{1}{3}(h_{i+1} - h_i) f_i^{(3)}$$

- Therefore the error has again increased to order $E(\Delta h)$ while with constant spacing it was $E(h^2)$.

GRID CONVERGENCE STUDIES FOR HURRICANE STORM SURGE COMPUTATIONS

- **Objectives**
  - Establish grid resolution requirements and the associated discretization error levels for computations with a range of storm parameters and shoreline geometries.
  - Develop graded grids which provide a consistent level of normalized extreme error throughout the hurricane.
  - Examine basic convergence behavior and study Richardson based error estimators

- Rectangular domain with shelf, slope and deep ocean bathymetries

- Forcing Function
  - Wind and pressure forcing from a synthetic hurricane as computed by the HURWIN model (Cardone, 1992) using a shore perpendicular track.

- Error Analysis for Hurricane Storm Surge Computations
  - Time history of extreme over- and under-prediction errors.
  - Apply Roache's (1994) Grid Convergence Index (GCI) to establish error bands
  - GCI's are based on Richardson extrapolation and computed using the difference between two discrete solutions at different spatial and/or temporal resolutions.
  - Comparing a coarse and a fine grid solution, the GCI's equal:

$$(GCI)^{Coarse} = F_S \frac{r^p|\varepsilon|}{r^p - 1} \quad \text{and} \quad (GCI)^{Fine} = F_S \frac{|\varepsilon|}{r^p - 1}$$

where

$\varepsilon \equiv f^{coarse} - f^{fine}$

r = refinement factor

p = order of the method

F_S = safety factor

- **Formal Convergence Rate**
  - **Determined by the leading order space and time truncation error terms**

- **Actual Asymptotic Convergence Rate**
  - **Interaction between the leading order and higher order truncation terms when response gradients increase with grid refinement**

$$\tau = \frac{1}{2!} h^2 \frac{d^3 f}{dx^3} + \frac{1}{3!} h^3 \frac{d^4 f}{dx^4} + \frac{1}{4!} h^4 \frac{d^5 f}{dx^5} + \ldots$$

Letting $\frac{d^3 f}{dx^3} = O\left(\frac{F}{h^3}\right)$

$$\tau = \frac{1}{2!}\frac{F}{h} + \frac{1}{3!}\frac{F}{h} + \frac{1}{4!}\frac{F}{h} + \ldots$$

- **Observed Convergence Rate**
  - **Often differs from Formal and Actual Asymptotic convergence rates due to:**
    - **Computer roundoff**
    - **"Relatively" coarse discretization leading to higher order truncation terms competing with the leading order truncation term**
    - **Space - time truncation error interaction**
    - **Parameter resolution effects**
    - **Frequency distribution of the response**
    - **Appearance of artificial flow features due to the discretization algorithm**
    - **Boundary condition implementation, placement or specification**

- **Convergence Study of Uniform Grids for an Idealized Rectangular Domain**

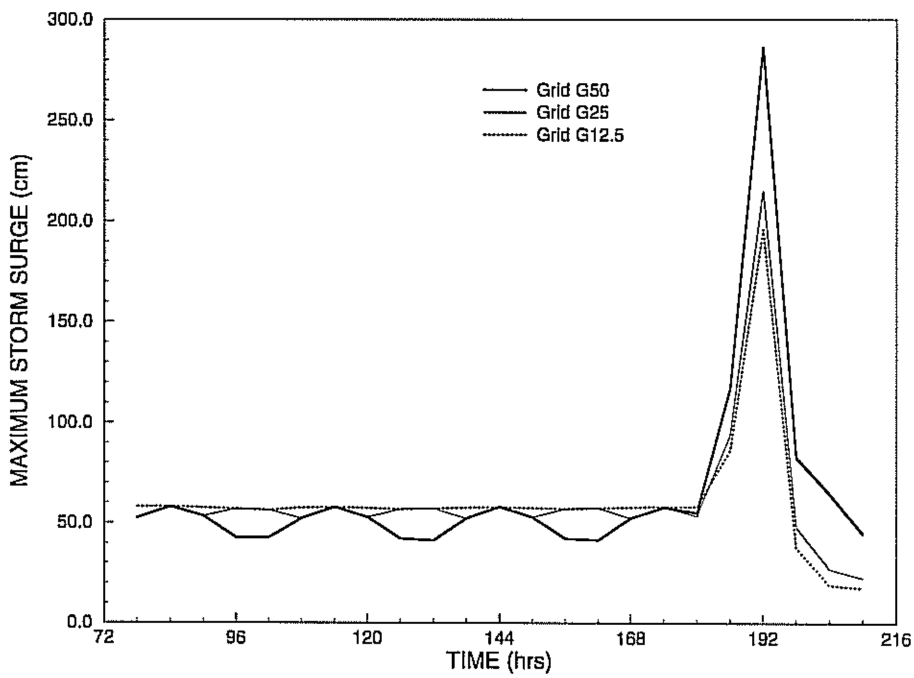
  - **Grids**

Grid	Structure	Grid Size (km)	Nodes
G50	uniform	50	3111
G25	uniform	25	12221
G12.5	uniform	12.5	48441

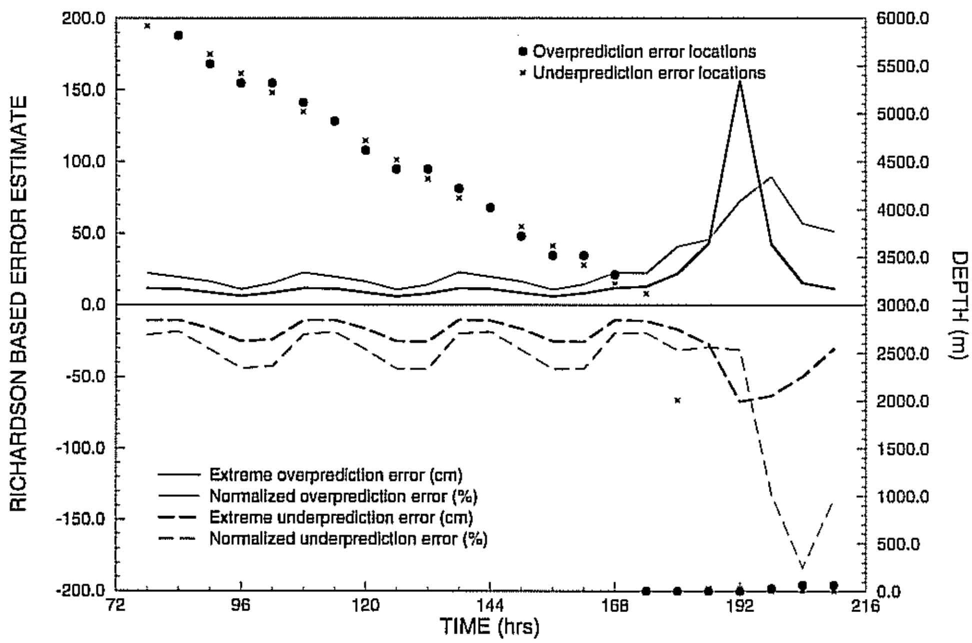
  - **Computed GCI's**

Grid	GCI Error Estimate
G01	$GCI^{Coarse}_{G50\text{-}G25}$
G02	$GCI^{Fine}_{G50\text{-}G25}$ and $GCI^{Coarse}_{G25\text{-}G12.5}$
G03	$GCI^{Fine}_{G25\text{-}G12.5}$

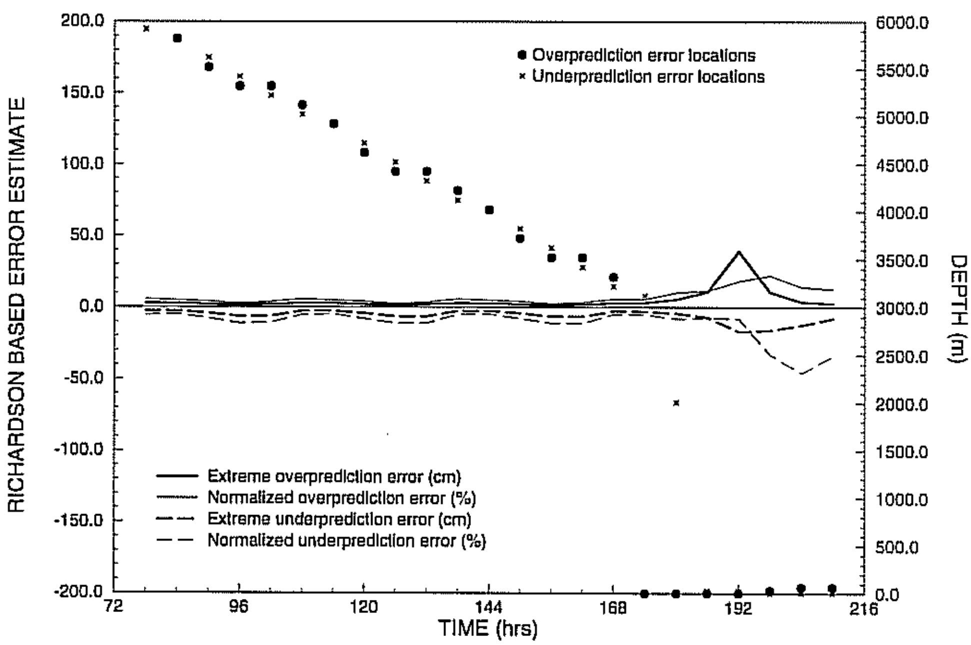
Time Histories of Peak Response for all Three Uniform Grids



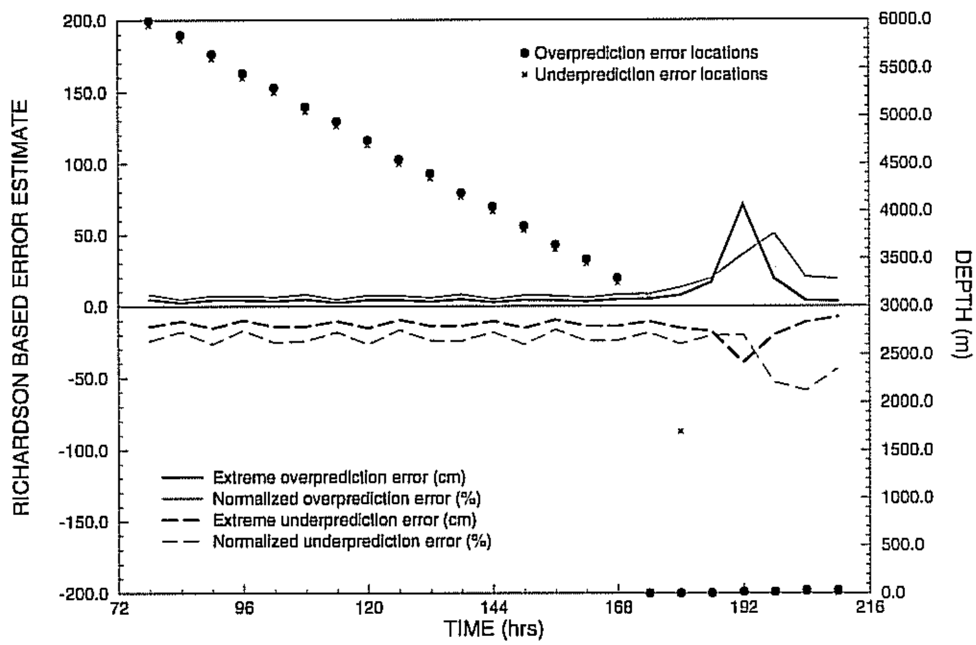
Richardson Based Error Estimates for Grid G50 - $GCI^{Coarse}_{G50\text{-}G25}$



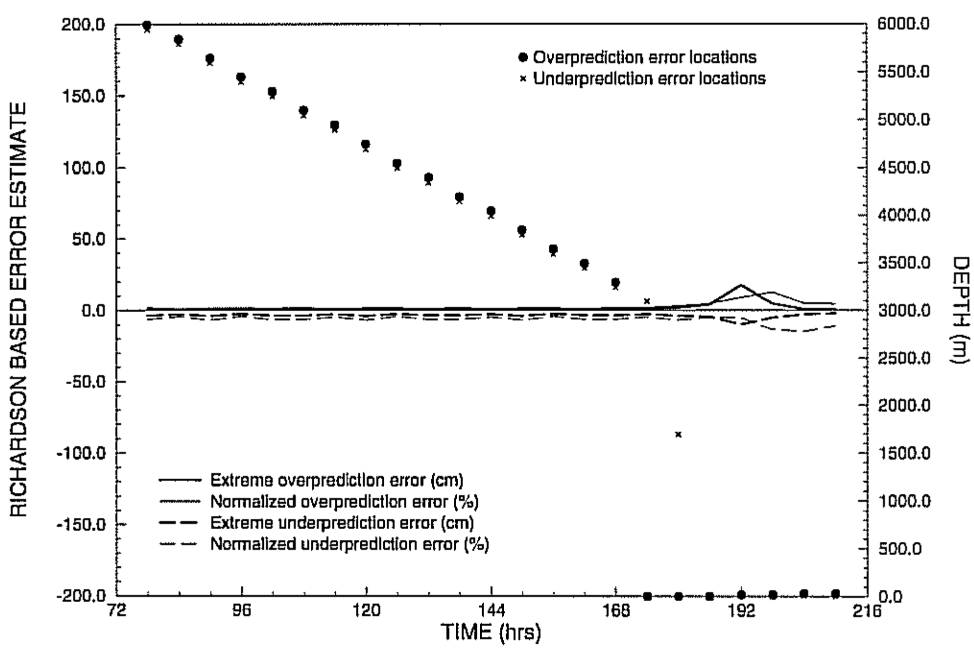
Richardson Based Error Estimates for Grid G25 Using Grid G50 - $GCI^{Fine}_{G50\text{-}G25}$



Richardson Based Error Estimates for Grid G25 Using Grid G12.5 - $GCI^{Coarse}_{G25-G12.5}$



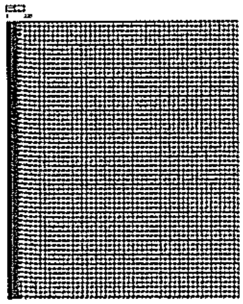
Richardson Based Error Estimates for Grid G12.5 Using Grid G25 - $GCI^{Fine}_{G25-G12.5}$

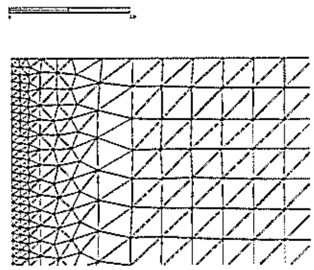


- **Convergence Study of a Graded Grid for an Idealized Rectangular Domain**
  - **Grids**

Grid	Structure	Grid Size (km)	Nodes
VG02	graded	12.5→50	4014
CVG02	graded	6.25→25	15647

  - **Grid VG02**

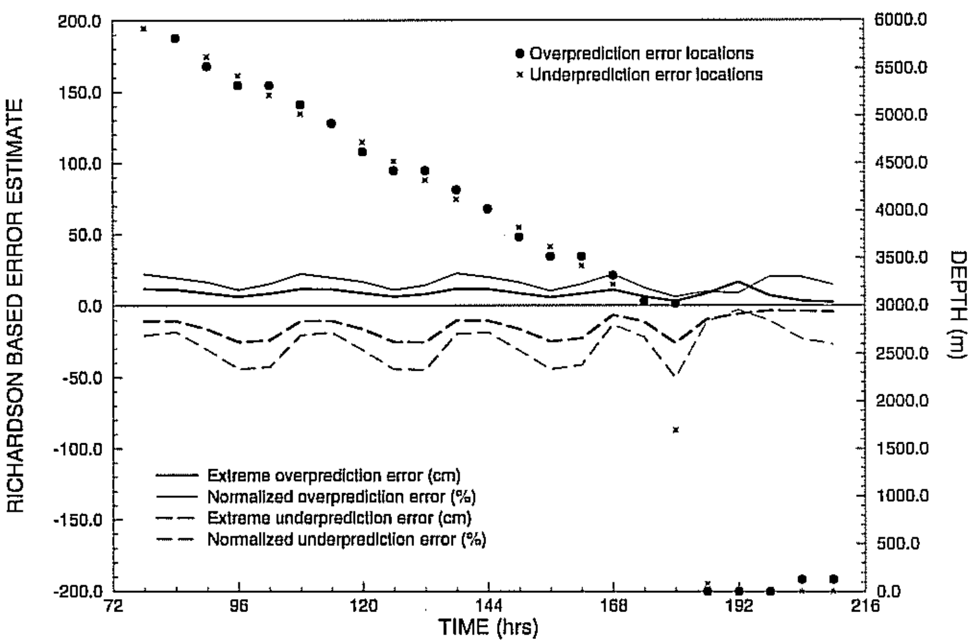




  - **Computed GCI's**

Grid	GCI Error Estimate
VG02	$GCI^{Coarse}_{VG02\text{-}CVG02}$

Richardson Based Error Estimates for Grid VG02 - $GCI^{Coarse}_{VG02\text{-}CVG02}$



- Results
  - Solutions converge monotonically with increasing grid refinement.
  - Actual convergence rate does not match formal rate, necessitating the use of a safety factor in error estimation.
  - The Richardson based GCI is a very useful tool for providing reasonable estimates for discretization errors.
  - Extensive studies varying domains, boundaries and storms have shown that grid resolution is the dominant factor in controlling normalized extreme error levels (normalized with peak surge).
  - While the storm is in deep water, under-resolution of the grid leads to dominant under-prediction of the inverted barometer due to aliasing of the pressure forcing function.
  - While the storm is on the continental shelf, under-resolution of the grid leads to a dominant over-prediction of the peak surge.