Lecture No. 5

Local Truncation Error:

Local Truncation error represents the difference between an exact differential equation and its FD representation at a point in space and time. Local truncation error provides a basis for comparing local accuracies of various difference schemes.

Example

Compute the local truncation error of the classical explicit difference approximat~~ely~~ion to:

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$$

FD representation is (explicit: forward in time; central in space):

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} = 0$$

This difference equation is satisfied for the numerical solution u only! ***It is <u>not</u> satisfied for the exact solution.***

The truncation error is represented by (where U = exact solution):

$$\tau_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} - \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{(\Delta x)^2}$$

We can work out the truncation error in more detail by Taylor series expanding:

$$U_{i+1,j} = U_{i,j} + \Delta x \left.\frac{\partial U}{\partial x}\right|_{i,j} + \frac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j} + \frac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 U}{\partial x^3}\right|_{i,j} + \dots$$

$$U_{i-1,j} = U_{i,j} - \Delta x \frac{\partial U}{\partial x}\bigg|_{i,j} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 U}{\partial x^2}\bigg|_{i,j} - \frac{1}{6}(\Delta x)^3 \frac{\partial^3 U}{\partial x^3}\bigg|_{i,j} + \ldots$$

$$U_{i,j+1} = U_{i,j} + \Delta t \frac{\partial U}{\partial t}\bigg|_{i,j} + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 U}{\partial t^2}\bigg|_{i,j} + \frac{1}{6}(\Delta t)^3 \frac{\partial^3 U}{\partial t^3}\bigg|_{i,j} + \ldots$$

Substituting in and re-arranging (by order of Δx, Δt) we have:

$$\tau_{i,j} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \frac{1}{2}\Delta t\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} - \frac{1}{12}\Delta x^2\left(\frac{\partial^4 U}{\partial x^4}\right)_{i,j}$$
$$+ \frac{1}{6}(\Delta t)^2\left(\frac{\partial^3 U}{\partial t^3}\right)_{i,j} - \frac{1}{360}(\Delta x)^4\left(\frac{\partial^6 U}{\partial x^6}\right) + \ldots$$

However since U is the exact solution, it identically satisfies the original p.d.e. at <u>all</u> points (i, j). Thus:

$$\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{i,j} = 0$$

Substituting into $\tau_{i,j}$ leads to:

$$\tau_{i,j} = \left(\frac{1}{2}\Delta t \frac{\partial^2 U}{\partial t^2} - \frac{1}{12}(\Delta x)^2 \frac{\partial^4 U}{\partial x^4}\right)_{i,j} + \left(\frac{1}{6}(\Delta t)^2 \frac{\partial^3 U}{\partial t^3} - \frac{1}{360}(\Delta x)^4 \frac{\partial^6 U}{\partial x^6}\right)_{i,j}$$

$$\Rightarrow$$

$$\tau_{i,j} = 0(\Delta t) + 0(\Delta x)^2$$

- Thus the classical explicit solution to $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ is $O(\Delta t)$ accurate in time and $O(\Delta x)^2$ accurate in space.

However we can make this method more accurate. We note from the p.d.e. that $\frac{\partial^2 U}{\partial t^2} = \frac{\partial^4 U}{\partial x^4}$. Substituting into the leading term in $\tau_{i,j}$:

$$\tau_{i,j} = \left(\frac{1}{2}\Delta t\frac{\partial^4 U}{\partial x^4} - \frac{1}{12}(\Delta x)^2\frac{\partial^4 U}{\partial x^4}\right)_{i,j} + 0(\Delta t)^2 + 0(\Delta x)^4$$

$$\Rightarrow$$

$$\tau_{i,j} = \frac{1}{2}\left(\frac{\partial^4 U}{\partial x^4}\right)_{i,j}\left(\Delta t - \frac{1}{6}(\Delta x)^2\right) + 0(\Delta t)^2 + 0(\Delta x)^4$$

If we let $\Delta t = \frac{1}{6}(\Delta x)^2$ then:

$$\tau_{i,j} = 0(\Delta t)^2 + 0(\Delta x)^4$$

- Thus we can make the classical explicit solution to $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ $O(\Delta t)^2$ accurate in time and $O(\Delta x)^4$ accurate in space ***if we select a very special relationship between Δx and Δt.***
- ***In this special case, the leading order space and time truncation errors cancel!***
- Recall that for stability we required $\Delta t < \frac{1}{2}(\Delta x)^2$ which was already making the time step too small. Therefore this is not a practical technique to enhance the order of accuracy in this particular case!

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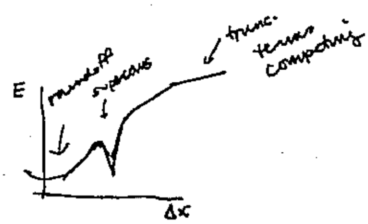


Accuracy of the C-N solutions to Linear Differential Equations

Let us formally investigate the time accuracy of the C-N scheme in conjunction with any linear operator in space using a truncation error analysis.

Let:

$$\frac{\partial u}{\partial t} = L(u)$$

where $L(u)$ = linear differential operator in space

Applying the C-N discretization in time:

$$\frac{u_{j+1} - u_j}{\Delta t} = \frac{1}{2}[L(u_{j+1}) + L(u_j)]$$

Let U = the exact solution at the nodes. Then τ_j, the local truncation error equals:

$$\tau_j = \frac{U_{j+1} - U_j}{\Delta t} - \frac{1}{2}[L(U_{j+1}) + L(U_j)]$$

Now we Taylor series expand for U_{j+1} about time level j:

$$U_{j+1} = U_j + \Delta t\frac{\partial U_j}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 U_j}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 U_j}{\partial t^3} + H.O.T.$$

Substituting for U_{j+1} into our FD expression:

$$\tau_j = \frac{1}{\Delta t}\left[U_j + \Delta t\frac{\partial U_j}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 U_j}{\partial t^2} + \frac{(\Delta t)^3}{6}\frac{\partial^3 U_j}{\partial t^3} - U_j\right]$$

$$-\frac{1}{2}\left[L\left(U_j + \Delta t\frac{\partial U_j}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 U_j}{\partial t^2} + \frac{(\Delta t)^3}{6}\frac{\partial^3 U_j}{\partial t^3}\right) + L(U_j)\right]$$

Since the operator $L(U)$ is linear we have:

$$\tau_j = \frac{1}{\Delta t}\left(\Delta t\frac{\partial U_j}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 U_j}{\partial t^2} + \frac{(\Delta t)^3}{6}\frac{\partial^3 U_j}{\partial t^3}\right) -$$

$$\frac{1}{2}\left(L(U_j) + \Delta t\frac{\partial}{\partial t}(L(U_j)) + \frac{(\Delta t)^2}{2}\frac{\partial^2}{\partial t^2}(L(U_j))\right.$$

$$\left. + \frac{(\Delta t)^3}{6}\frac{\partial^3}{\partial t^3}(L(U_j)) + L(U_j)\right)$$

Now grouping terms and pulling out derivatives we have:

$$\tau_j = \left[\frac{\partial U_j}{\partial t} - L(U_j)\right] + \frac{\Delta t}{2}\frac{\partial}{\partial t}\left[\frac{\partial U_j}{\partial t} - L(U_j)\right]$$

$$+ \frac{\Delta t^2}{6}\frac{\partial^2}{\partial t^2}\left[\frac{\partial U_j}{\partial t} - \frac{3}{2}L(U_j)\right]$$

$$+ \frac{\Delta t^3}{24}\frac{\partial^3}{\partial t^3}\left[\frac{\partial U_j}{\partial t} - 2L(U_j)\right] + H.O.T.$$

Thus the local truncation error may be represented by

$$\tau_j = \frac{\Delta t}{2}\frac{\partial}{\partial t}\left[\frac{\partial U_j}{\partial t} - L(U_j)\right] + \frac{(\Delta t)^2}{12}\frac{\partial^2}{\partial t^2}\left[\frac{\partial U_j}{\partial t} - \frac{3}{2}L(U_j)\right] + H.O.T.$$

However since the first term identically satisfies the original p.d.e. it must equal zero!

$$\tau_j = \frac{(\Delta t)^2}{12}\frac{\partial^2}{\partial t^2}\left[\frac{\partial U_j}{\partial t} - \frac{3}{2}L(U_j)\right]$$

- ***Therefore C-N applied to linear differential operators always results in 2nd order time accuracy.***
- This analysis also shows that the accuracy to which each term in the p.d.e. is evaluated does not necessarily reflect the actual accuracy of the scheme. The actual accuracy is dependent on exactly how both the time and space discretizations were implemented. In addition both space and time truncation errors from any of the terms in the p.d.e. can and do interact.

Consistency (or Compatibility) of a Numerical Discretization

A consistent numerical scheme is one which converges to the solution of the p.d.e. being discretized. Thus as mesh length tends to zero, truncation terms also tend to zero.

Example of a Consistent Solution

For the explicit solution to the diffusion equation, the truncation error equals:

$$\tau_{i,j} = \frac{1}{2}\Delta t\frac{\partial^2 U}{\partial t^2} - \frac{1}{12}(\Delta x)^2\frac{\partial^4 U}{\partial x^4} + 0(\Delta t)^2 + 0(\Delta x)^4$$

Thus as $\Delta x \to 0$ and $\Delta t \to 0 \quad \Rightarrow \quad \tau_{i,j} \to 0$

Inconsistent/Incompatible Scheme:

For an inconsistent numerical scheme, the numerical solution converges to the solution of a <u>different</u> p.d.e.!

Example of a Conditionally Consistent Solution

Consider:

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$$

discretized central in time and central in space as:

$$\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t} - \frac{u_{i+1,j} - 2\{\theta u_{i,j+1} + (1-\theta) u_{i,j-1}\} + u_{i-1,j}}{\Delta x^2} = 0$$

The truncation error equals:

$$\tau_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} - \frac{U_{i+1,j} - 2\{\theta U_{i,j+1} + (1-\theta) U_{i,j-1}\} + U_{i-1,j}}{\Delta x^2}$$

TS expanding for $U_{i,j+1}$, $U_{i,j-1}$, $U_{i+1,j}$ and $U_{i-1,j}$ and then substituting and re-arranging, we have:

$$\tau_{i,j} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \left(\frac{(\Delta t)^2}{6}\frac{\partial^3 U}{\partial t^3} - \frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^4}\right.$$
$$\left. + (2\theta - 1)\frac{2\Delta t}{(\Delta x)^2}\frac{\partial U}{\partial t} + \frac{(\Delta t)^2}{(\Delta x)^2}\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + 0\left(\frac{\Delta t^3}{\Delta x^2}, (\Delta x)^4, (\Delta t)^4\right)$$

The first term in $\tau_{i,j}$ satisfies the original p.d.e. Thus:

$$\tau_{i,j} = \left(\frac{(\Delta t)^2}{6}\frac{\partial^3 U}{\partial t^3} - \frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^4} + (2\theta - 1)\frac{2\Delta t}{(\Delta x)^2}\frac{\partial U}{\partial t} + \frac{(\Delta t)^2}{(\Delta x)^2}\frac{\partial^2 U}{\partial t^2}\right)_{i,j} +$$
$$+ O\left(\frac{\Delta t^3}{\Delta x^2}, (\Delta x)^4, (\Delta t)^4\right)$$

- ***In this case we do not necessarily have*** $\tau_{i,j} \to 0$ ***as*** $\Delta x \to 0$ ***and*** $\Delta t \to 0$.

- Therefore, we must consider how fast Δx and Δt decrease to zero <u>relative</u> to each other.

<u>Case 1</u>: Let $\Delta t = r\Delta x$ where $r =$ positive constant:

$$\tau_{i,j} = \frac{r^2(\Delta x)^2}{6}\frac{\partial^3 U}{\partial t^3} - \frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^4} + (2\theta - 1)\frac{2r}{\Delta x}\frac{\partial U}{\partial t} + r^2\frac{\partial^2 U}{\partial t^2}$$

$$+ 0\left(r^3\Delta x,\ (\Delta x)^4,\ (r\Delta x)^4\right)$$

as $\Delta x \to 0$

$$\tau_{i,j} = (2\theta - 1)\frac{2r}{\Delta x}\frac{\partial U}{\partial t} + r^2\frac{\partial^2 U}{\partial t^2}$$

- When $\theta \neq \frac{1}{2}$, the first term blows up.
- When $\theta = \frac{1}{2}$, the second term makes numerical solution consistent with p.d.e.:

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + r^2\frac{\partial^2 U}{\partial t^2} = 0$$

- Therefore this numerical scheme and selection of Δt and Δx are inconsistent with our originally stated p.d.e.

<u>Case 2</u>: Let $\Delta t = r(\Delta x)^2$

$$\tau_{i,j} = \frac{r^2(\Delta x)^4}{6}\frac{\partial^3 U}{\partial t^3} - \frac{(\Delta x)^2}{12}\frac{\partial^4 U}{\partial x^4} + (2\theta - 1)\frac{2r(\Delta x)^2}{(\Delta x)^2}\frac{\partial U}{\partial t} + \frac{r^2(\Delta x)^4}{\Delta x^2}\frac{\partial^2 U}{\partial t^2}$$

$$+ 0\left(\frac{r^3(\Delta x)^6}{(\Delta x)^2},\ (\Delta x)^4,\ r^4(\Delta x)^8\right)$$

Letting $\Delta x \to 0$

$$\tau_{i,j} = (2\theta - 1)\, 2r\frac{\partial U}{\partial t}$$

- When $\theta \neq \frac{1}{2}$, this numerical solution is inconsistent with the originally stated p.d.e. and is consistent with the p.d.e.

$$\{1 + 2(2\theta - 1)r\}\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$$

- When $\theta = \frac{1}{2}$, the scheme is consistent. The Du Fort and Frankel 3 level explicit scheme is stable for all r and is of $0(\Delta t^2, \Delta x^2)$.
- In general the scheme presented in this example is conditionally consistent, and must satisfy the conditions $\theta = \frac{1}{2}$ and $\Delta t = r\Delta x^2$ to be consistent.

Summary of Accuracy Considerations:

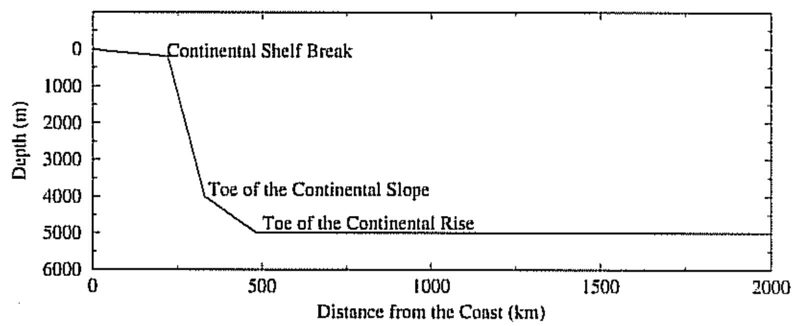
- The order of accuracy of a scheme must be determined by examining the difference equation as a whole, not the individual component terms.
- We can play with things such as the relationship between Δt and Δx to in certain cases increase the order of accuracy of the scheme.
- ***Not all numerical discretizations are consistent with the p.d.e. that you're trying to solve. THIS IS VERY IMPORTANT TO REALIZE!!***

GRID DESIGN BASED ON LOCAL TRUNCATION ERROR ANALYSIS

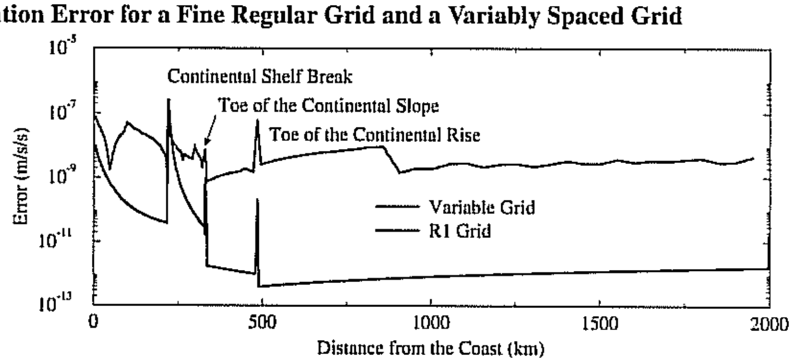
- **Problem**
  - **The** $\frac{\lambda}{\Delta x} = \frac{\sqrt{gh}\ T}{\Delta x}$ **criterion can result in poor grids for tidal computations.**
  - **The $\lambda/\Delta x$ criterion does not indicate a need for high resolution in the vicinity of steep topographic gradients such as the continental shelf break and continental slope.**
  - **The $\lambda/\Delta x$ criterion does not recognize two-dimensional structure in the response.**

- **Concept**
  - **Formally compute local truncation error.**
  - **Compute nodal spacing such that the estimated local truncation error is kept constant throughout the domain.**
  - **Response derivative terms in the expression for truncation error are estimated using either fine or coarse regular grid computations.**

One Dimensional Idealized Bathymetry



Truncation Error for a Fine Regular Grid and a Variably Spaced Grid



- **Truncation error analysis for the momentum equation results in:**
  - **For irregularly spaced grids:**

$$\tau_{ME} = \left(\frac{i\omega+\tau}{6}\right)\left[2(\Delta x_{i+1}-\Delta x_i)\frac{\partial \hat{u}_j}{\partial x} + \left(\frac{\Delta x_i^3+\Delta x_{i+1}^3}{\Delta x_i+\Delta x_{i+1}}\right)\frac{\partial^2 \hat{u}_j}{\partial x^2}\right.$$

$$\left. + \frac{1}{3}(\Delta x_{i+1}-\Delta x_i)(\Delta x_i^2+\Delta x_{i+1}^2)\frac{\partial^3 \hat{u}_j}{\partial x^3} + \frac{1}{12}\left(\frac{\Delta x_i^5+\Delta x_{i+1}^5}{\Delta x_i+\Delta x_{i+1}}\right)\frac{\partial^4 \hat{u}_j}{\partial x^4}\right]$$

$$+ \frac{g}{6}\left[3(\Delta x_{i+1}-\Delta x_i)\frac{\partial^2 \hat{\eta}_j}{\partial x^2} + \left(\frac{\Delta x_i^3+\Delta x_{i+1}^3}{\Delta x_i+\Delta x_{i+1}}\right)\frac{\partial^3 \hat{\eta}_j}{\partial x^3}\right.$$

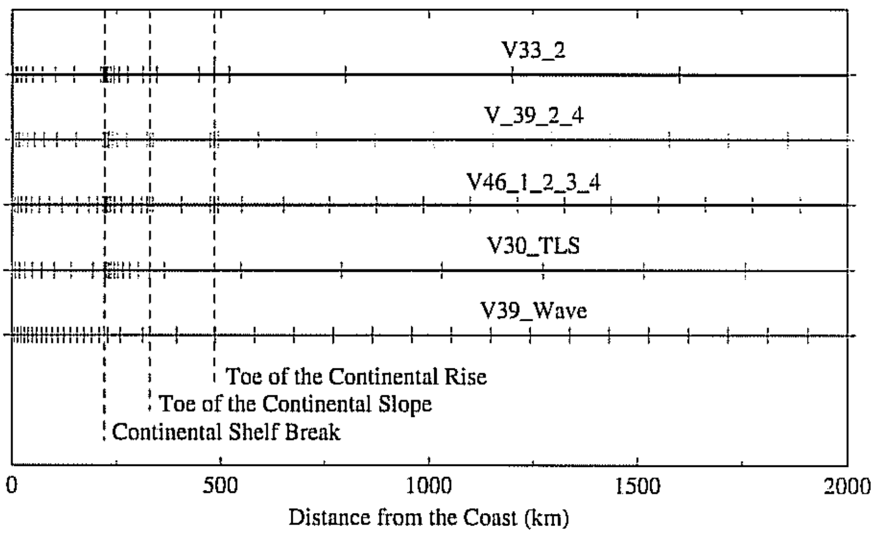
$$\left. + \frac{1}{4}(\Delta x_{i+1}-\Delta x_i)(\Delta x_i^2+\Delta x_{i+1}^2)\frac{\partial^4 \hat{\eta}_j}{\partial x^4} + \frac{1}{20}\left(\frac{\Delta x_i^5+\Delta x_{i+1}^5}{\Delta x_i+\Delta x_{i+1}}\right)\frac{\partial^5 \hat{\eta}_j}{\partial x^5}\right] + \text{H.O.T.}$$

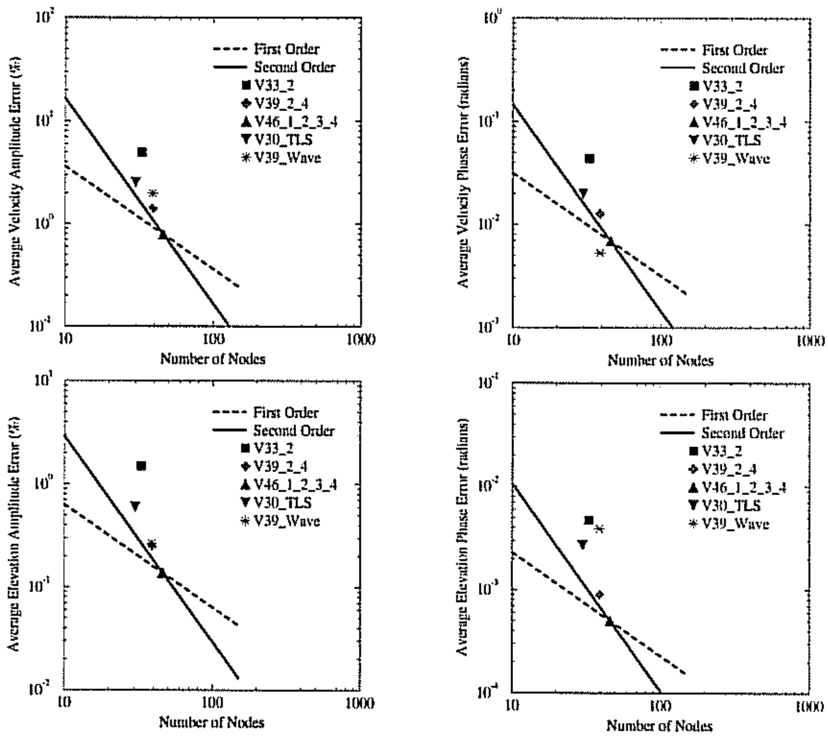
  - **For regular grids with** $\Delta x_i = \Delta x_{i+1} = \Delta x$**:**

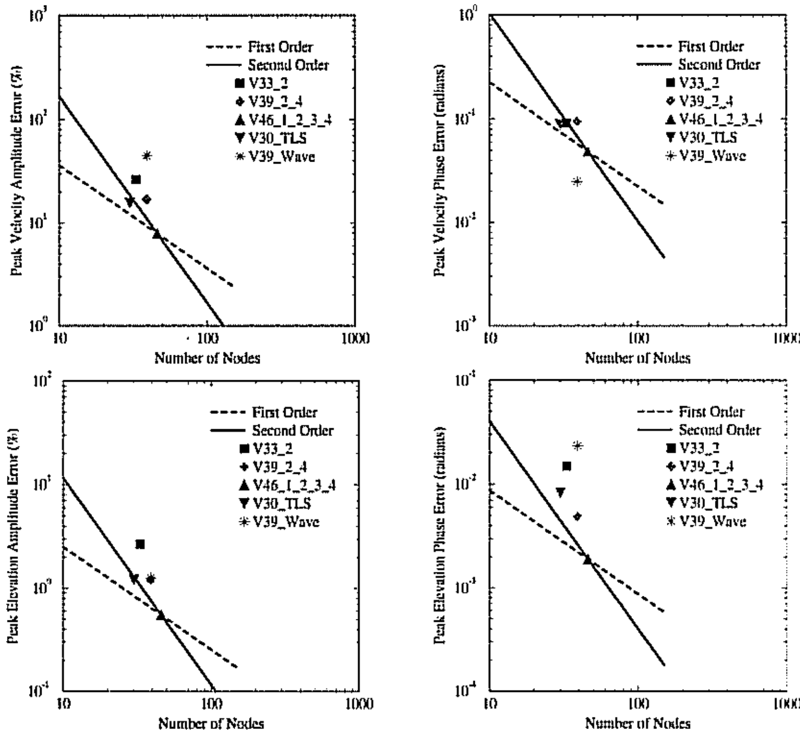
$$\tau_{ME} = \frac{i\omega+\tau}{6}\left(\Delta x^2\frac{\partial^2 \hat{u}_j}{\partial x^2} + \frac{\Delta x^4}{12}\frac{\partial^4 \hat{u}_j}{\partial x^4}\right) + \frac{g}{6}\left(\Delta x^2\frac{\partial^3 \hat{\eta}_j}{\partial x^3} + \frac{\Delta x^4}{20}\frac{\partial^5 \hat{\eta}_j}{\partial x^5}\right) + \text{H.O.T.}$$

- **Compare grids designed with the following criteria:**
  - **Limit the *Second order* truncation error terms in the expression for τ_{ME} with constant Δx**
    - **Note that we neglect adjacent element size variability**
  - **Limit the *Second and Fourth order* truncation error terms in the expression for τ_{ME} with constant Δx**
    - **Note that we neglect adjacent element size variability**
  - **Limit the *First through Fourth order* truncation error terms in the general expression for τ_{ME} with variable Δx_i**
    - **Note that we fully consider adjacent element size variability**
  - **Topographic Length Scale (TLS) Criterion:** $\Delta x \le \dfrac{\alpha h}{\left(\dfrac{\partial h}{\partial x}\right)}$
    - **Note that in the limit as topography flattens,** $\frac{\partial h}{\partial x} \to 0$, **element size increases,** $\Delta x \to \infty$
  - **Wavelength to grid size criterion** $\dfrac{\lambda}{\Delta x} = \dfrac{\sqrt{gh}\ T}{\Delta x} \ge 100$
    - **Note that the 100 value is high relative to typical discretizations used in most studies**

Resulting Grids Produced with the Various Criteria







- **Results**
  - **Truncation error based grids attempt to provide an even spatial distribution of the source of the error, the local truncation errors in the Momentum and GWCE equations.**
  - **Truncation error based grids provide excellent results on a per node basis compared to the widely used wavelength to grid size and even TLS criterion.**
  - **Fourth order truncation terms should be considered in regions with large grid spacing (i.e. deep waters).**
  - **Odd order truncations terms associated with the rate of change of grid size can be important in the local truncation error. This limits the rate of adjacent element expansion.**
  - **Resulting grids indicate high resolution requirements over the continental shelf, shelf break and slope.**
  - **Derivatives in truncation error terms can be estimated using both fine and coarse grids**

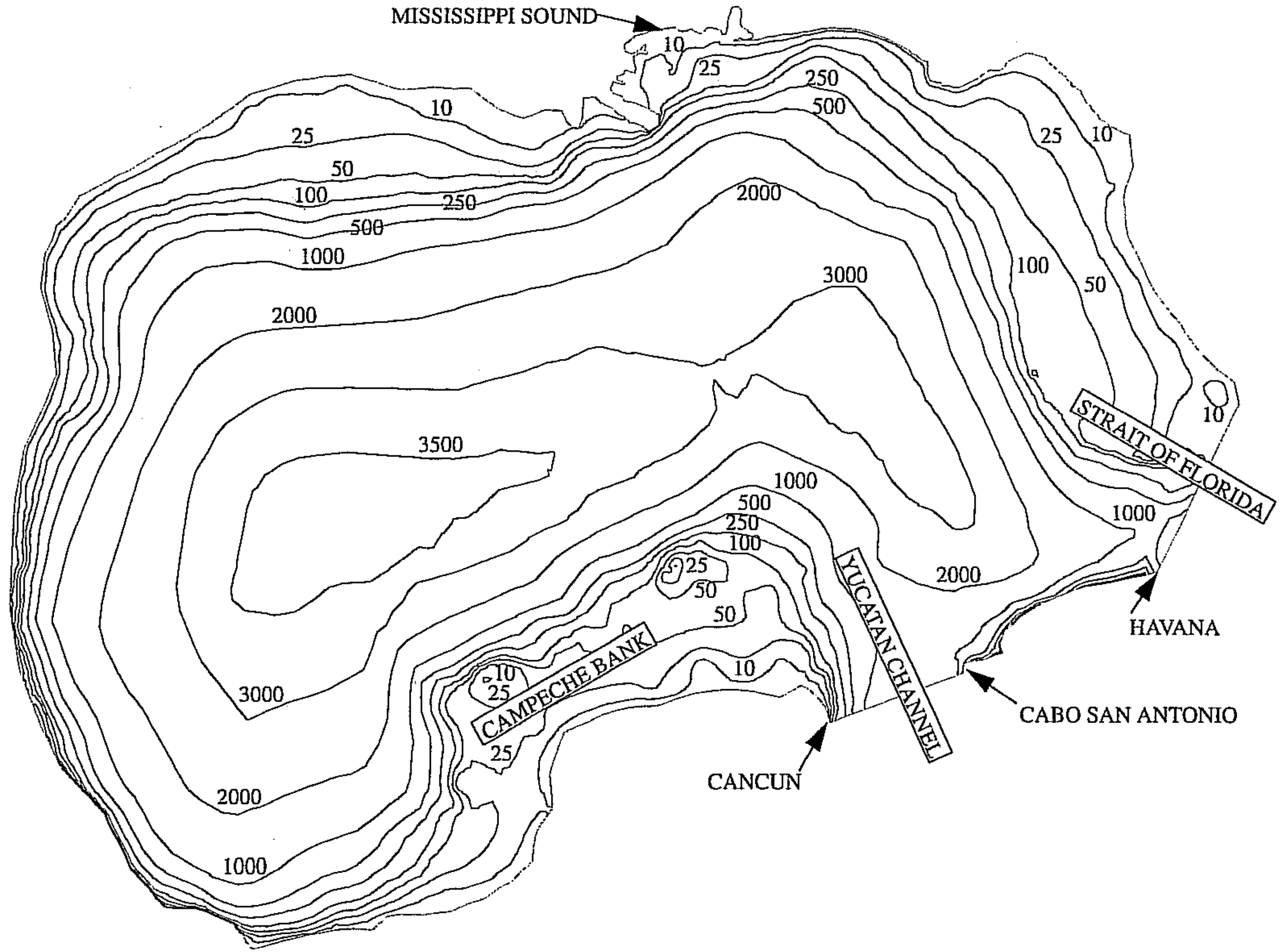


Figure 1. Gulf of Mexico model domain with bathymetric contours (meters).

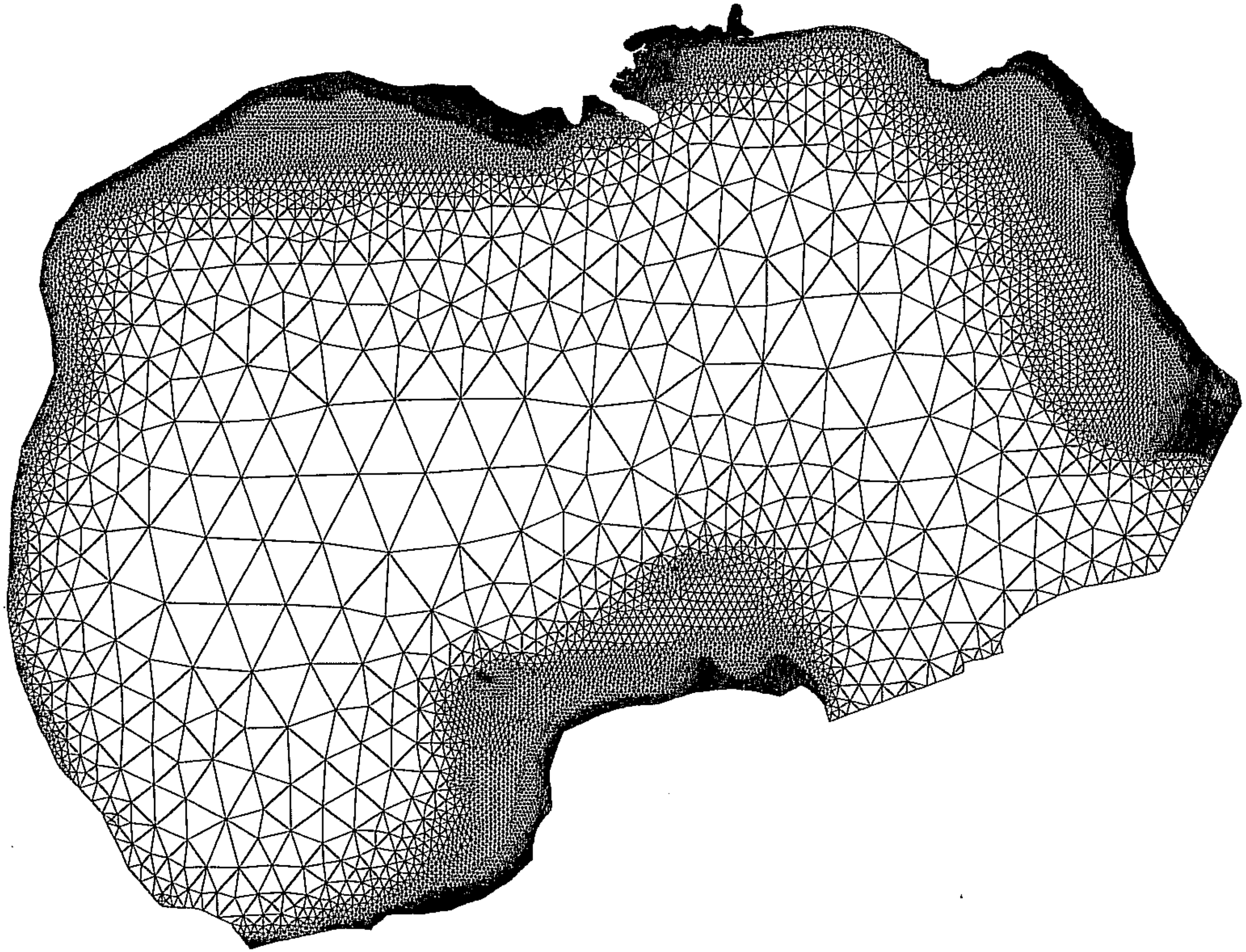


Figure 2. GOMEX_W100: Wavelength-based graded mesh for the Gulf of Mexico.

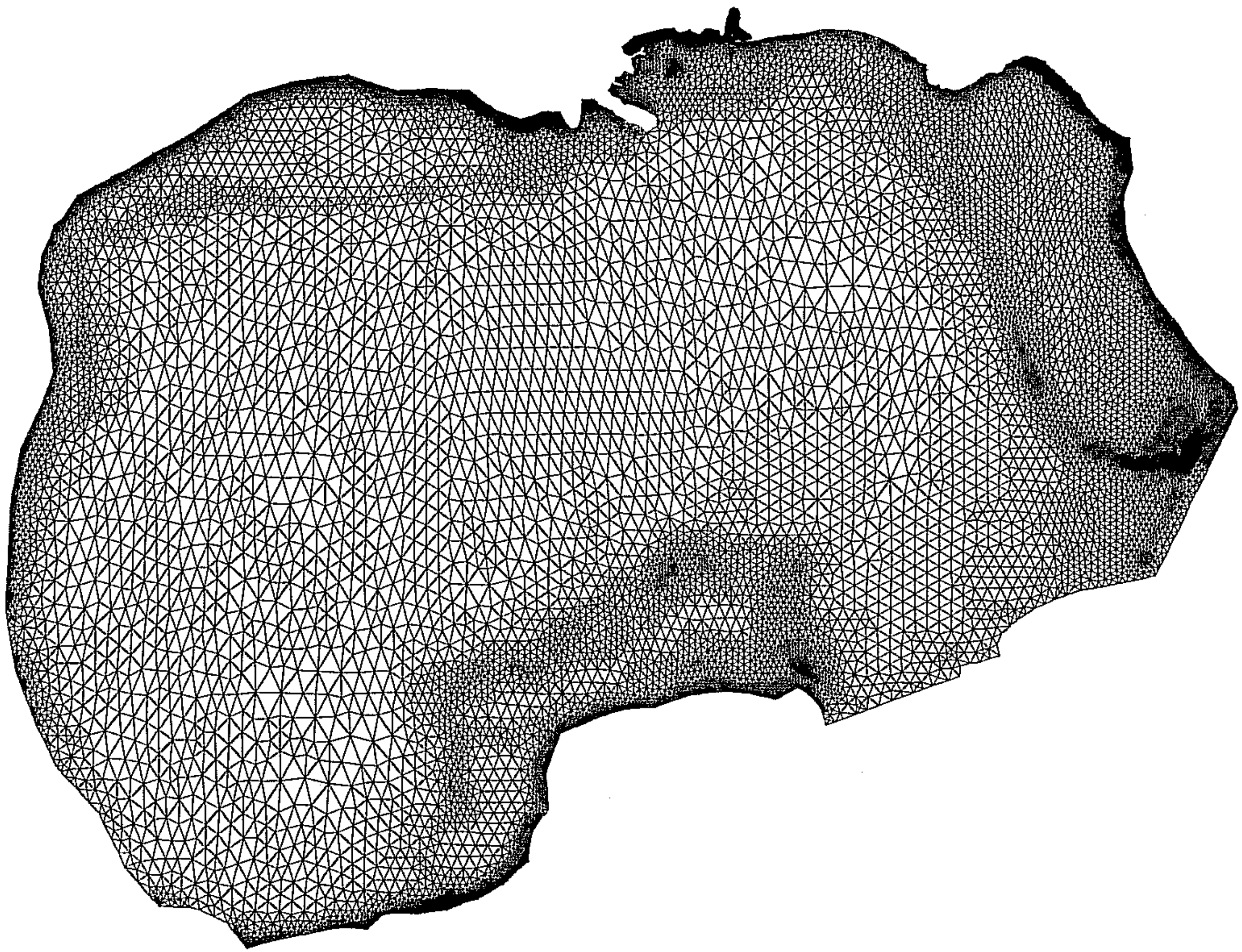


Figure 7. An LTEA-based finite element grid for the Gulf of Mexico.

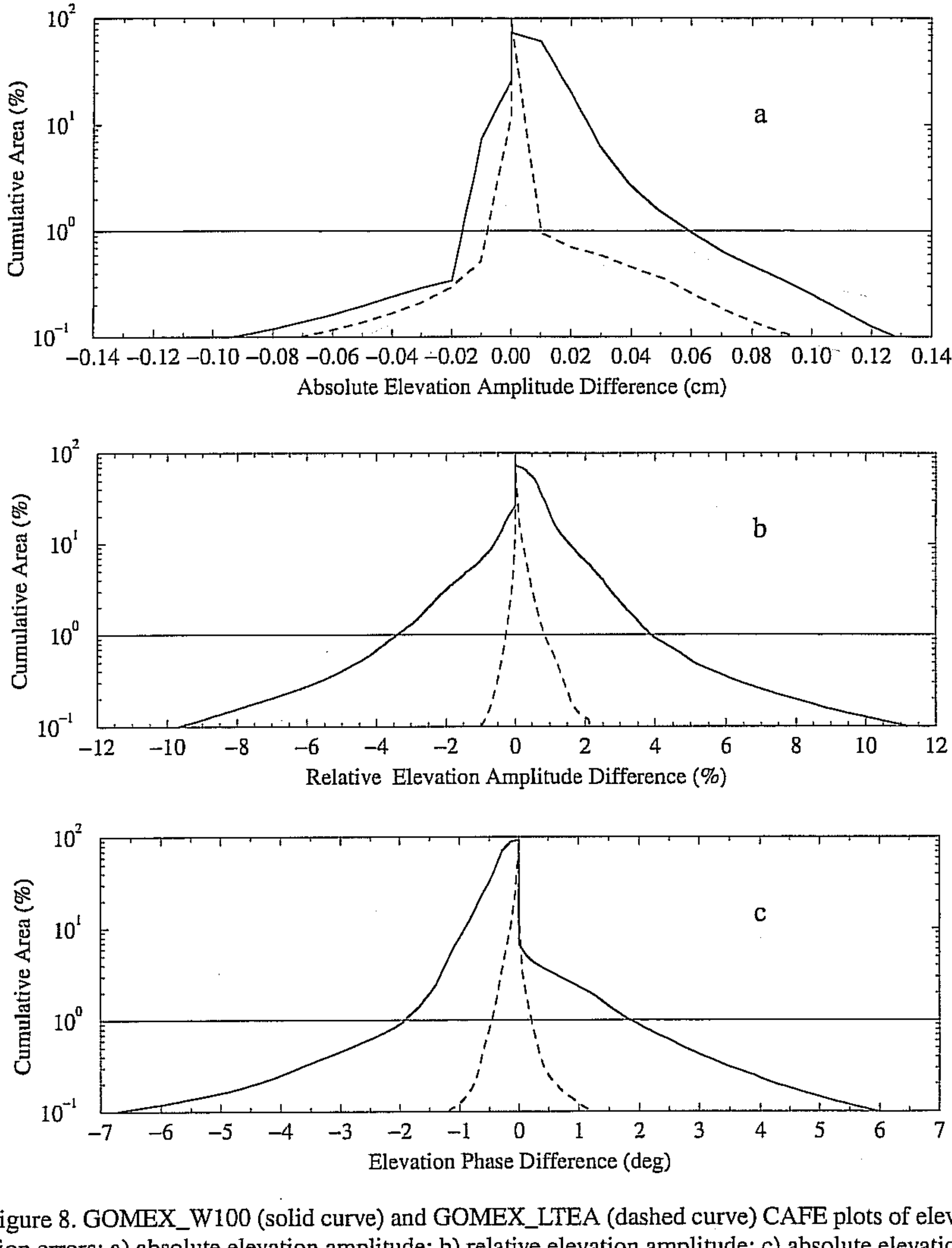


Figure 8. GOMEX_W100 (solid curve) and GOMEX_LTEA (dashed curve) CAFE plots of elevation errors: a) absolute elevation amplitude; b) relative elevation amplitude; c) absolute elevation phase.

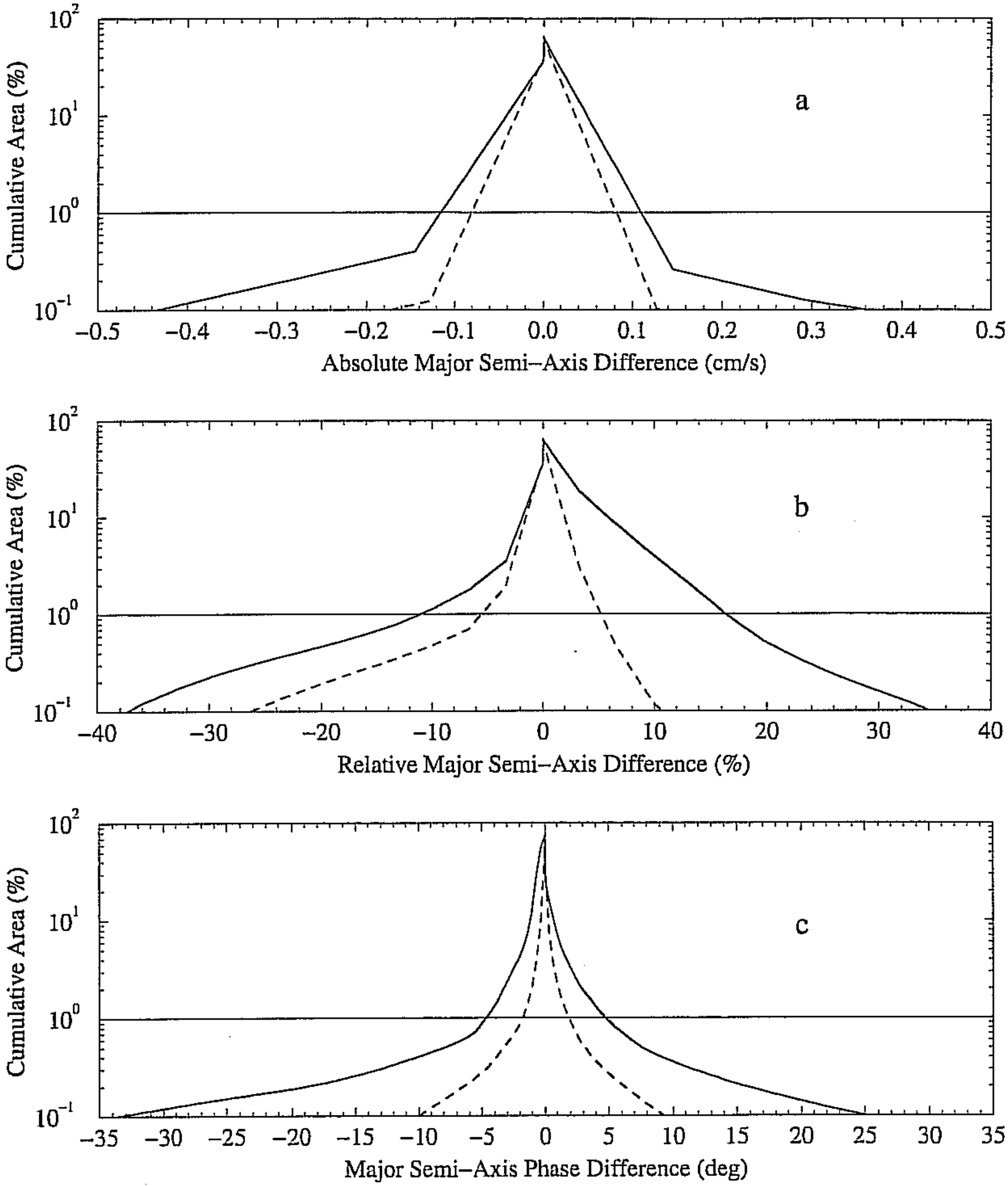


Figure 9. GOMEX_W100 (solid curve) and GOMEX_LTEA (dashed curve) CAFE plots of velocity errors: a) absolute major semi-axis; b) relative major semi-axis; c) absolute major semi-axis phase.

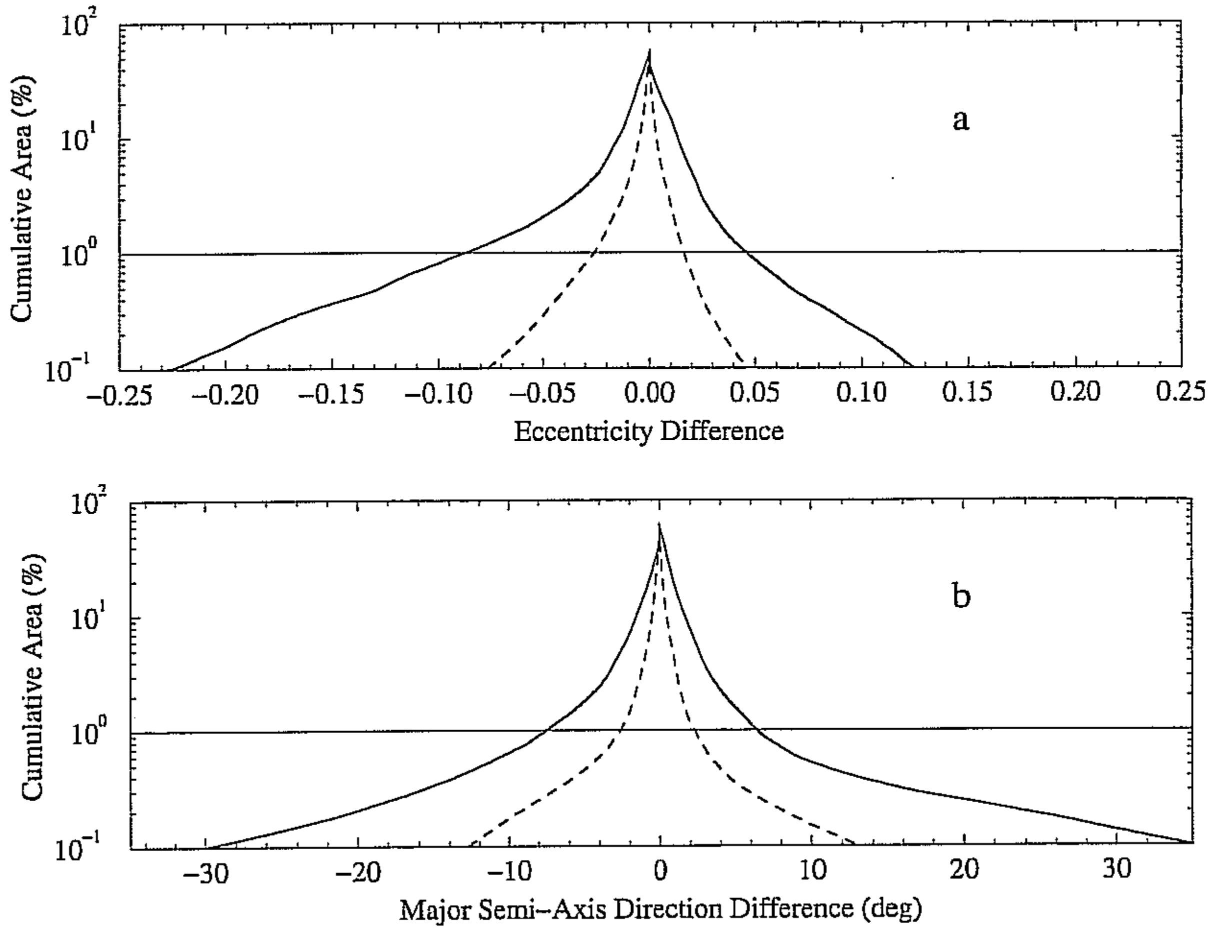


Figure 10. GOMEX_W100 (solid curve) and GOMEX_LTEA (dashed curve) CAFE plots of velocity errors: a) absolute eccentricity; b) absolute major semi-axis direction.

TABLE 1. ELEVATION ERROR MEASURES, RELATIVE TO THE SPLIT-BY-FOUR VERSION OF THE LTEA-BASED GRID

	GOMEX_LTEA	GOMEX_W100
Elevation Amplitude (absolute)		
Percent area exceeding -0.01 cm	**0.54%**	7.27%
Percent area exceeding +0.01 cm	**1.04%**	60.07%
Total percent area exceeding ±0.01 cm	**1.58%**	67.34%
Elevation Amplitude (relative)		
Percent area exceeding -1.0%	**0.10%**	6.54%
Percent area exceeding +1.0%	**0.71%**	21.13%
Total percent area exceeding ±1.0%	**0.81%**	27.67%
Elevation Phase (°)		
Percent area exceeding -1.0°	**0.12%**	7.99%
Percent area exceeding +1.0°	**0.12%**	2.34%
Total percent area exceeding ±1.0°	**0.24%**	10.33%

TABLE 2. VELOCITY ERROR MEASURES, RELATIVE TO THE SPLIT-BY-FOUR VERSION OF THE LTEA-BASED GRID

	GOMEX_LTEA	GOMEX_W100
Major Semi-Axis (absolute)		
Percent area exceeding -0.1 cm/s	**0.42%**	1.64%
Percent area exceeding +0.1 cm/s	**0.40%**	1.44%
Total percent area exceeding±0.1 cm/s	**0.82%**	3.08%
Major Semi-Axis (relative)		
Percent area exceeding -5%	**1.14%**	2.52%
Percent area exceeding +5%	**1.14%**	12.26%
Total percent area exceeding ± 5%	**2.28%**	14.78%
Major Semi-Axis Phase (°)		
Percent area exceeding -2.0°	**0.85%**	4.15%
Percent area exceeding +2.0°	**0.92%**	3.30%
Total percent area exceeding ±2.0 °	**1.77%**	7.45%
Eccentricity		
Percent area exceeding -0.04	**0.48%**	2.71%
Percent area exceeding +0.04	**0.14%**	1.26%
Total percent area exceeding ±0.04	**0.62%**	3.97%
Major Semi-Axis Direction (°)		
Percent area exceeding -5.0°	**0.45%**	1.78%
Percent area exceeding +5.0°	**0.34%**	1.58%
Total percent area exceeding ±5.0 °	**0.79%**	3.36%