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I. Derivation of error evaluation formula

Let's assume that we have implemented a method/code that is n^{th} order accurate in space

$$f = \tilde{f}_h + A h^n + H.O.T.$$

where

f = the exact solution
$\tilde{f}_h$ = the numerical solution with grid spacing h
h = the grid spacing
n = the spatial order of the method
A = a coefficient that includes ~~the~~ a m^{th} order derivative of the solution BUT is grid spacing independent
H.O.T. = higher order terms

We are going to assume that we are within what is refered to as the assymptotic range – that the leading order truncation term truly dominates

We now consider two grid spacings h_1 and h_2 with $h_1 > h_2$
We compute solutions for each grid spacing using our code

$$h_1 \rightarrow \tilde{f}_{h_1}$$
$$h_2 \rightarrow \tilde{f}_{h_2}$$

Now we note that:

$$f = \tilde{f}_{h_1} + A h_1^n$$
$$f = \tilde{f}_{h_2} + A h_2^n$$

task list:

Considering f and A as the unknowns, we solve for A

$$A = \frac{\tilde{f}_{h_2} - \tilde{f}_{h_1}}{h_1^n - h_2^n}$$

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Thus the error for the coarse grid solution is

$$E_{h_1} = A h_1^n$$

$$= \frac{\tilde{f}_{h_2} - \tilde{f}_{h_1}}{h_1^n - h_2^n} \cdot h_1^n$$

$$= \frac{\left(\tilde{f}_{h_2} - \tilde{f}_{h_1}\right)\left(\frac{h_1}{h_2}\right)^n}{\left(\frac{h_1}{h_2}\right)^n - \left(\frac{h_2}{h_2}\right)^n}$$

$$\boxed{E_{h_1} = \frac{\varepsilon\, r^n}{r^n - 1}}$$

where $\varepsilon \equiv \left|\tilde{f}_{h_2} - \tilde{f}_{h_1}\right|$

$r \equiv \frac{h_1}{h_2}$ = refinement factor

n = spatial order of the method

h_1 = coarse grid spatial resolution

h_2 = fine grid spatial resolution

Similarly we can show that the fine grid solution estimated error is

$$\boxed{E_{h_2} = \frac{\varepsilon}{r^n - 1}}$$

task list:

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II How to estimate spatial errors – practical implementation guide

We will assume a 2^{nd} order method / code in space and time (for example a Crank-Nicolson - central in space solution)

STEP A: Compute solutions with the following space and time steps

$$h_1 \,;\, \Delta t_1 \rightarrow \tilde{f}_{h_1;\Delta t_1}$$

$$h_1 \,;\, \Delta t_2 = \frac{\Delta t_1}{2} \rightarrow \tilde{f}_{h_1;\Delta t_2}$$

$$h_2 = \frac{h_1}{2} \,;\, \Delta t_2 = \frac{\Delta t_1}{2} \rightarrow \tilde{f}_{h_2;\Delta t_2}$$

Note that you should reduce time step proportionally to your space step if the time and space discretizations are equal order (in this case 2^{nd}) approximations

STEP B: Estimate the differences between your solutions on the shared nodes of the coarse grid (i.e. project your fine grid solution onto the coarse grid).
We will estimate the differences between solutions as a root mean square.
If nx and ny are your coarse grid discretization number of nodes in the x and y directions respectively

task list:

$$\varepsilon_{(h_1;\Delta t_1 \text{ to } h_1;\Delta t_2)} = \sqrt{\frac{1}{nx*ny+1}\sum_{i=1}^{nx}\sum_{j=1}^{ny}\left|\tilde{f}_{h_1;\Delta t_1}(i,j) - \tilde{f}_{h_1;\Delta t_2}(i,j)\right|^2}$$

$$\varepsilon_{(h_1;\Delta t_1 \text{ to } h_2;\Delta t_2)} = \sqrt{\frac{1}{nx*ny+1}\sum_{i=1}^{nx}\sum_{j=1}^{ny}\left|\tilde{f}_{h_1;\Delta t_1}(i,j) - \tilde{f}_{h_2;\Delta t_2}(2*i-1, 2*j-1)\right|^2}$$

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STEP C : Ensure that the spatial error dominates the temporal error

$$\text{If} \quad \mathcal{E}_{(h_1; \Delta t_1 \text{ to } h_1; \frac{\Delta t_1}{2})} \ll \mathcal{E}_{(h_1; \Delta t_1 \text{ to } h_2; \Delta t_2)}$$

then spatial error dominates

If this is not the case, go back to step A and reduce Δt_1

STEP D : Estimate spatial errors

Coarse Grid Error :

$$E_{h_1 : h_1 - h_2} = \frac{\mathcal{E}_{(h_1; \Delta t_1 \text{ to } h_2; \Delta t_2)}}{2^2 - 1} \; 2^2$$

Fine Grid Error :

$$E_{h_2 : h_1 - h_2} = \frac{\mathcal{E}_{(h_1; \Delta t_1 \text{ to } h_2; \Delta t_2)}}{2^2 - 1}$$

STEP E: You may want to ensure that you are indeed in the so called "assymptotic" range - i.e. that you have sufficient resolution that the leading order truncation term truly dominates

You can do this by introducing a resolution

$$h_3 = \frac{h_2}{2} = \frac{h_1}{4}$$

Now you can compare grid h_2 to grid h_3 and compare errors:

If you compare the errors for the h_2 grid two different ways

$$E_{h_2 : h_1 - h_2} \simeq E_{h_2 : h_2 - h_3}$$

then you are in the assymptotic range and your solution is indeed second order accurate in space

task list: